Quivers

When working with algebras it is convenient to understand the concepts of quivers and relations.

Definition 1. A quiver Q is a directed graph $Q = (Q_0, Q_1, t, h)$ where:

- Q_0 - set of vertices
- Q_1 - set of arrows
- $t : Q_1 \to Q_0$ assigns the tail of the arrow
- $h : Q_1 \to Q_0$ assigns the head of the arrow

We now wish to understand representations of quivers.

Definition 2. A representation $\underline{V}$ of a quiver Q, over a field k, assigns a vector space $V(i)$ to each $i \in Q_0$ and a transformation $\varphi_\alpha : V(i) \to V(j)$ for each $\alpha \in Q_1$.

Path Algebras

A path from e to f, for $e, f \in Q_0$, is a sequence of the form $(e \mid \alpha_1 \mid \cdots \mid \alpha_l \mid f)$ with $\alpha_i \in Q_1$ and

$$\overset{e}{\bullet} \xleftarrow{\alpha_1} \bullet \xleftarrow{\alpha_2} \bullet \cdots \bullet \xleftarrow{\alpha_l} \overset{f}{\bullet}$$

We denote a path of length zero by $(e \mid \ \mid e)$.

The *path algebra* kQ of the quiver Q is defined as the k-vector space with basis Q_1. The product in kQ is by path concatenation, when it makes sense, and zero otherwise.

Remark 1. **Fact:** kQ is finite-dimensional if and only if Q_0 and Q_1 is finite and there are no cyclic paths.

Theorem 1 (Gabriel)**.** *Any basic finite-dimensional k-algebra A, is of the form kQ/I for a unique quiver Q and ideal I.*

Basic Algebras

Definition 3. We say that a finite-dimensional k-algebra A, is a *basic algebra* if all simple A-modules are 1-dimensional.

Here are some remarks:

Remark 2. Given any arbitrary finite-dimensional k-algebra A, the quiver Q describing the basic algebra may be found as follows:

(1) Q_0 is identified with the collection of isomorphism types of simple A-modules.
(2) Q_1 the arrows $i \to j$ in number is given by $\dim_k \operatorname{Ext}^1(s_i, s_j)$.

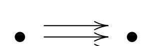
Representation Type

Suppose k is an infinite field and A is a finite-dimensional k-algebra. Then A

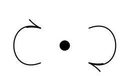
(1) is of *finite representation type* if there are only finitely many isomorphism classes of indecomposable A-modules.
(2) is of *tame representation type* if the indecomposable A-modules can be parameterized by a 1-paraameter subgroup.
(3) is of *wild representation type* if its module category is comparable with $k\langle x, y\rangle$-Mod.

Here are some examples of quivers:

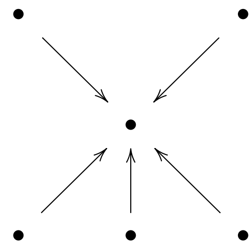
Example 1. Quiver with three arrows



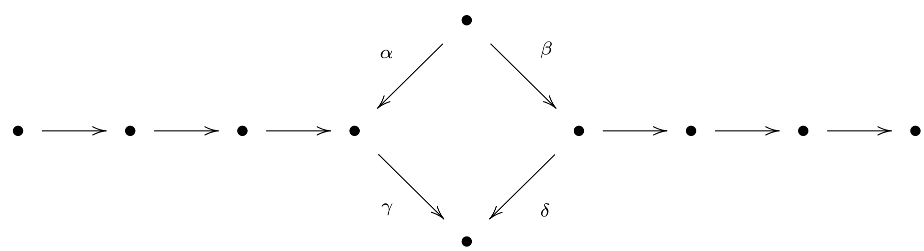
Example 2. Two-loop Quiver



Example 3. Five subspace Quiver



Example 4.



Let $I = \langle \alpha\gamma - \beta\delta \rangle$. The algebra $\frac{kQ}{I}$ is wild.

Theorem 2 (Drozd, Crawley-Boevey)**.** *Over an algebraically closed field, every finite-dimensional algebra is of finite type, tame, or wild, and these types are mutually exclusive.*

Theorem 3 (Baur, Kokorin, Mart'yanov)**.** *The theory of $k\langle x, y\rangle$-modules of finite-dimension is undecidable.*

Here are some properties:

If A is a finite-dimensional k-algebra, then

(1) If I is some ideal such that A/I is wild, then A is wild.

(2) If e is an idempotent such that eAe is wild, then A is wild.

(3) A is wild if and only if its basic algebra is wild.

(4) If A is wild, then A^{op} is wild.

Theorem 4 (Bondarenko, Drozd, Ringel)**.** *Let G be a finite group of order $p^\alpha q$ and k an infinite field of chracteristic p. If B is a block of kG and D a defect group of B, then B*

(1) *is of finite type if D is cyclic.*

(2) *is of tame type if $p = 2$ and D is a Sylow 2-subgroup of G that is*

- *Dihedral*

$$\langle r, s \mid r^n = s^2 = 1, srs = r^{-1} \rangle$$

- *Semidihedral*

$$\langle r, s \mid r^{2^{n-1}} = s^2 = 1, srs = r^{2^{n-2}} - 1 \rangle$$

- *Generalized Quaternion*

$$\langle x, y \mid x^{2n} = y^4 = 1, x^n = y^2, y^{-1}xy = x^{-1} \rangle.$$

Example 5. Consider an elementary 2-group $E = (\mathbb{Z}/2)^r$, and k an infinite field of chracteristic 2. Then

(1) kE is of finite type if and only if $E = \mathbb{Z}/2$.

(2) kE is of tame type if and only if $E = \mathbb{Z}/2 \oplus \mathbb{Z}/2$

(3) kE is of wild type otherwise.