Characters

For some field k, we want to consider maps, $\chi : G \to k$ that are *class functions.* Note that any homomorphism $\chi : G \to k^*$ is a class function, that is constant on conjugacy classes, because k^* is abelian. In which case, we say that $\chi : G \to k^*$ is a *linear character.*

Example 1. Consider an n-dimensional algebraic torus $X = (\mathbb{C}^*)^n$. Then we're considering all of the algebraic group morphisms $\chi : X \to \mathbb{C}^*$, gives a free abelian $\mathbb{Z}$-module of rank n.

Example 2. We can also consider $Y = \mathrm{GL}_n(V)$ and all of the algebraic group morphisms $\chi : Y \to \mathbb{C}^*$. For example, $\det : \mathrm{GL}_n(\mathbb{C}^*)$. We denote the character group Y^*.

Remark 1. If $\chi : G \to \mathbb{C}^*$ is a linear character, then $\mathrm{Tr}(g) = \mathrm{Tr}(\chi(g)) = \chi(g)$.

Definition 1. Let G be a finite group and $\rho : G \to \mathrm{GL}(V)$ be a representation. We say that

$$\begin{aligned} \chi_V : G &\longrightarrow k \\ g &\longmapsto \mathrm{Tr}(g) = \mathrm{Tr}(\rho(g)) \end{aligned}$$

is a *character* of the group G afforded by the representation.

Definition 2. Let G be a finite group of $\#G = p^\alpha q$ and k of chracteristic p.

(1) An element $g \in G$ is said to be *p-regular* if $(|g|, p) = 1$.

(2) An element $g \in G$ is said to be *p-singular* if it's not p-regular.

Remark 2.

(1) If $m := \#\langle g \rangle = p^\alpha q$ with $p \nmid q$ and $1 = ap^\alpha + bq$, where $a, b \in \mathbb{Z}$. We say that

$$g_{p'} = g^{ap^\alpha} \quad \text{and} \quad g_p = g^{bq}$$

are the *p-regular part* and *p-part* of g, respectively.

(2) $g = g^{ap^\alpha + bq} = g^{ap^\alpha} \cdot g^{bq} = g^{bq} g^{p^\alpha}$ and one sees that $g = g_{p'} g_q = g_q g_{p'}$.

(3) We set $G_{p'}$ to be the set of p-regular elements of G. In this vein, we say that a conjugacy class is p-regular or p-singular according to its elements. We denote the p-regular conjugacy classes by $\mathrm{Cl}(G_{p'})$.

Lemma 1. *Let k be an algebraically closed field of chracteristic p and G a finite group of $\#G = p^\alpha q$. Then for all $\chi \in \mathrm{Char}_k(G)$*

$$\chi(g) = \chi(g_{p'}).$$

Proof. Suppose that $G = \langle g \rangle$. Let $\rho : G \to \mathrm{GL}(V)$ be a representation and χ_V the chracter afforded by ρ. Note that $\rho(g) = \rho(g_{p'})\rho(g_p)$. We may choosed a basis so that $\rho(g_{p'})$ and $\rho(g_p)$ are triangular matrices and $\rho(g_p)$ is unipotent, and thus only 1's on the diagonal. So, the diagonal elements of $\rho(g)$ and $\rho(g_{p'})$ coincide. $\square$

Brauer Characters

Let p be a prime and $m \in \mathbb{N}$. Define

$$U_{p'} = \langle \{\zeta_\alpha \mid p \nmid \alpha\} \rangle \quad \text{and} \quad U_{p',m} = \{\zeta \in U_{p'} \mid \zeta^m = 1\}$$

where $\zeta_k = e^{2\pi i/\alpha}$.

Let G be a finite group with exponent $m = p^r q$, with $p \nmid q$, and $\mathbb{F}$ a field of chracteristic p with algebraic closure $\overline{\mathbb{F}}$. Suppose that W is an $\mathbb{F}G$-module with $\dim_{\mathbb{F}} W = n$, and $\rho : G \to \mathrm{GL}(W)$ a representation. Consider a ring homomorphism $\theta : \mathbb{Z}[\zeta_\alpha] \to \overline{\mathbb{F}}$. For $g \in G_{p'}$ the eigenvalues of $\rho(g)$ are mth roots of unity in $\overline{F}$, and take the form $\theta(\zeta_1(g)), \dots, \theta(\zeta_n(g))$, where $\zeta_i(g) \in U_{p',m}$ are uniquely determined. We now define

$$\phi_W : G_{p'} \to \mathbb{C}$$

$$g \mapsto \sum_i \zeta_i(g)$$

and call this the Brauer chracter affored by W.

Definition 3.

(1) If W is irreducible we say that ϕ_W is irreducible.

(2) The set of irreducible Brauer chracters is denoted by $\mathrm{IBr}(G)$.

Remark 3. **Fact:** If $p \mid \#G$, then $\mathrm{IBr}(G) = \mathrm{Irr}(G)$.

p-Modular Systems

Let $\rho : G \to \mathrm{GL}_n(\mathbb{Q})$ be a representation. There is an equivalent representation $\rho_0 : G \to \mathrm{GL}_n(\mathbb{Z})$, which one can easily extend via the natural projection, to a representation over $\mathbb{F}_p$:

$$\overline{\rho_0} : \pi_p \circ \rho_0 : G \to \mathrm{GL}_n(\mathbb{F}_p)$$

where $\pi_p : \mathbb{Z} \to \mathbb{Z}/p$. We want to understand how we can link modular representations and ordinary representations. We have an embedding and a projection

$$\mathbb{Q}G \hookleftarrow \mathbb{Z}G \twoheadrightarrow \mathbb{F}_pG$$

which we wish to generalize.

Let R be a valuation ring, $\mathfrak{m}$ the unique maximal ideal, and quotient field

$$K = R \cup \{x^{-1} \mid x \in R - \{0\}\} \neq R$$

(1) We're assuming R is an integral domain.

(2) R is local

(3) R is integrally closed in K

(4) Any finitely generated torsion-free R-module is free.

Theorem 1. *If R is a valuation ring with quotient field K, any representation $\rho : G \to \mathrm{GL}_n(K)$ is equivalent to some representation $\rho_0 : G \to \mathrm{GL}_n(K)$, where $\rho_0(g) \in \mathrm{GL}_n(R)$ for all $g \in G$.*

If K is a field of chracteristic zero and p is a prime, then there is a valuation ring R with quotient field K and char $R/J(R) = p$.

We now define a *p-modular system.* If p is a prime, (K, R, F) is call a *p-modular system* if R is a complete discrete valuation ring with coefficient field K of chracteristic zero and $R/\pi R \cong \mathbb{F}$ is of chracteristic p.

One says that (K, R, F) is a *splitting system* if K contains a splitting field for $x^e - 1$, where e is the exponent of G.