Algebras

Let R be a ring with 1_R such that one can write $1_R = e_1 + \cdots + e_j$, where the e_i are primitive central orthogonal idempotents.

- **Primitive:** Can't be written as a sum of idempotents.
- **Central:** Idempotent is in the center of the ring.
- **Orthogonal:** $e_i e_j = 0 = e_j e_i$ for all $i \neq j$.

This gives us a decomposition into blocks:

$$R = e_1 R \oplus e_2 R \oplus \cdots \oplus e_j R$$

$$= B_1 \oplus B_2 \oplus \cdots \oplus B_j.$$

If M is an R-module, then we can decompose it as a direct sum:

$$M = e_1 M \oplus e_2 M \oplus \cdots e_j M = M_1 \oplus \cdots \oplus M_j.$$

We say that M_i *belongs to* or *lies in* the block B_i.

Now, suppose that ${}_R M$ is indecomposable. Then M can be decomposed as above, but no submodule of M has a direct complement. Hence $M = e_i M$ for some i and $e_k M = 0$, for all $k \neq i$.

- The indecomposable M belongs to the block B_i.
- The indecomposables (and simples) are classified into blocks.

Proposition 1. *Let A be a finite-dimensional k-algebra, k algebraically closed, and $Z(A)$ the center of A. Then there is a bloack decomposition*

$$Z(A) = e_1 Z(A) \oplus e_2 Z(A) \oplus \cdots \oplus e_j Z(A)$$

where each $e_i Z(A)$ is loacal and there is an inclusion map

$$\frac{e_i Z(A)}{J(e_i Z(A))} \hookrightarrow \mathrm{End}_A(V)$$

for each indecomposable A-module V in B_i.

Proof. A decomposition of 1_A as a sum of central idempotents in A and in $Z(A)$ are the same thing. Hence, the block decomposition of $Z(A)$ is as stated. The $Z(A)$ is commutative so,

$$e_i Z(A) \cong \mathrm{End}_{Z(A)}(e_i Z(A)) \cong e_i Z(A) e_i$$

which is local as a consequence of the Krull-Schmidt theorem.

If V is a simple A-module lying in B_i, then $e_i Z(A)$ acts on V nontrivially as endomorphisms, because e_i acts as the identity. Hence,

$$e_i Z(A) \to \mathrm{End}_A(V)$$

and since $e_i Z(A)$ is a local division ring we get the induced injection stated. □

Group Algebras

We now wish to specialize to group algebras. Recall that

$$kG = \left\{ \sum_{g \in G} \alpha_g \cdot g \mid \alpha_g \in k \right\}$$

The key idea for what we want to do is regard kG as a module for the group algebra $k[G \times G]$.

Q: How do we do this?

A: We need to define an action

$$k[G \times G] \times k[G] \longrightarrow k[G]$$

$$(g, h) \times a \longmapsto gah^{-1}$$

Remark 1. **Fact:** The $k[G \times G]$-submodules of kG are the ideals of kG. In particular, we can decompose kG as a direct sum of $k[G \times G]$-modules:

$$kG = B_1 \oplus \cdots \oplus B_k$$

the blocks of kG.

Let $\delta : G \to G \times G$ so that $g \mapsto (g, g)$ be the diagonal map. If $H, K \leq G$ with δH and δK conjugate in $G \times G$, then H and K are conjugate in G. That is, there exists $(a, b) \in G \times G$ such that

$$(a, b)\delta H(a^{-1}, b^{-1}) = \delta K$$

projecting onto the first coordiante one sees that H and K are conjugate in G.

Theorem 1. *Let U be an indecomposable kG-module.*

(1) *There is a p-subgroup of G, call it Q, unique up to conjugacy in G, such that U is relatively H-projective, for a subgroup H of G if and only if H contaings a conjugate of Q.*

(2) *There is an indecomposable kQ-module S, such that U is (isomorphic with) a direct summand of $S^G = k[G] \otimes_{kQ} S$.*

We now make the definitions so the theorem can make some kind of remote sense:

Definition 1.

(1) We say that U is *relativel H-projective* if $\phi : V \twoheadrightarrow U$ of kG-modules and ϕ split as a kH-homomorphism implies that ϕ is split.

(2) We say that Q, as in the theorem, is a *vertex* of U and S is a *source* of U.

We are now at the point where we can define what a defect group is:

Theorem 2. *If B is a block of kG, then B has a vertex, as a $k[G \times G]$-module, of the form δD, where D is a p-subgroup of G.*

Defect Groups

We have decomposed kG into blocks $kG = B_1 \oplus \cdots \oplus B_j$ as $k[G \times G]$-modules.

Remark 2. Given a block B_i, there exists a p-subgroup $D \leq G$ such that δD is a vertex of B_i, as a $k[G \times G]$-module. This D is the *defect group* of B_i.

Theorem 3 (Brauer's First Main Theorem)**.** *If D is a p-subgroup of G and H is a subgroup of G containing $N_G(D)$, then there exists a one-to-one correspondence between the blocks of H with defect group D and the blocks of G with defect group D by letting the block b of H correspond to the block b^G of G.*

Remark 3.

(1) We say that b is the *Brauer Correspondent* of b^G.

(2) Determining the blocks of G with defect group D is a "local" question:

It's enough to answer the question for $N_G(D)$.

Theorem 4 (Blocks of Defect Zero)**.** *If B is a block of kG with defect group D, then the following are equivalent:*

(1) $J(B) = 0$ *implies that B is a full matrix ring over a division ring.*

(2) *D is the trivial subgroup of G.*

(3) *B contains a projective simple module.*