Suzuki Groups ${}^2B_2(q^2)$

The Suzuki groups of Lie type are obtained by considering symplectic groups and a generalized Frobenius map. Our setup is as follows:

- $\Bbbk = \overline{\mathbb{F}}_2$
- G is the 4-dimensional symplectic group over $\Bbbk$

$$\mathrm{Sp}_4(\Bbbk) = \left\{ A \in \mathrm{Mat}_{4\times 4} \mid A^T J A = J \right\}$$

  where

$$J = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

- Define

$$\varphi : \mathrm{GL}_4(\Bbbk) \to \mathrm{GL}_4(\Bbbk)$$
$$A \mapsto J(A^T)^{-1}J$$

Then one sees that the fixed point set $(\mathrm{GL}_4(\Bbbk))^\varphi$ is a subgroup of $\mathrm{Sp}_4(\Bbbk)$. That is, $\varphi(A) = A$ and $\varphi(A) = J(A^T)^{-1}J$ so

$$\begin{aligned} A &= J(A^T)^{-1}J \\ A^T J &= JA^{-1}J \\ A^T J &= JA^{-1} \\ A^T J A &= J \end{aligned}$$

and therefore $A \in \mathrm{Sp}_4(\Bbbk)$. In fact, $(\mathrm{GL}_4(\Bbbk))^\varphi \cong \mathrm{Sp}_4(\Bbbk)$.

Define

$$H := \{h(t,u) \mid t,u \in \Bbbk^*\}$$

where

$$h(t,u) = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & u^{-1} & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix}$$

Furthermore, $N = H \cdot W$ where the Weyl group W is generated by

$$s_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } s_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Considering that the only nonzero entries of $s_\alpha s_\beta$ occur as dot products we obtain the permutation matrix

$$s_\alpha s_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

which corresponds to the cycle $\sigma = (1243)$ which has order 4.

The elements s_α and s_β are involutions and $s_\alpha s_\beta$ has order 4. We divide a square into its chambers

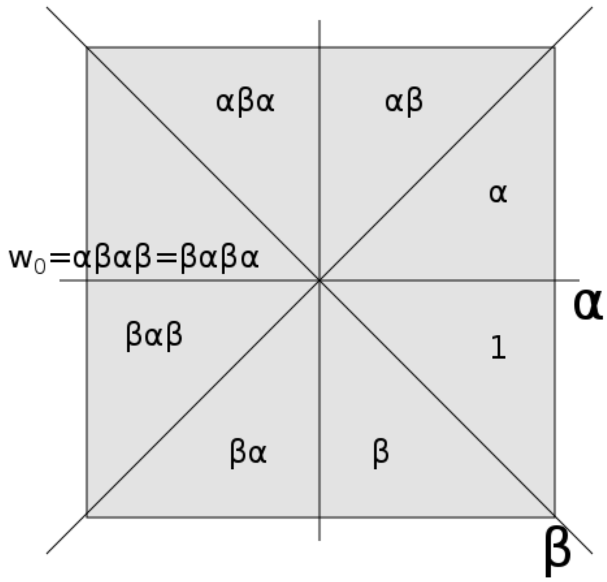

FIGURE 1. Weyl Chambers

The element w_0 is the *unique* longest word in the Coxeter group W. Furthermore, each of the chambers above corresponds to an element of the group. We have

- Coxeter system of type B_2 where $S = \{s_\alpha, s_\beta\}$
- $\langle s_\alpha, s_\beta \mid (s_\alpha s_\beta)^4 = s_\alpha^2 = s_\beta^2 = 1\rangle$
- $\mathbb{Z}/4 \rtimes \mathbb{Z}/2 \cong D_8$

We now take a look at some elements of G:

$$x_{alpha}(t) = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_{\beta}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x_{\alpha+\beta}(t) = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_{2\alpha+\beta}(t) = \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $t \in \Bbbk$. Define $x_{-r}(t) = x_r(t)^T$ for $r \in \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta\}$ and

$$\Phi := \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\}$$

Remark 1. (1) The set Φ is a root systme of type $B_2 = C_2$, where α and β are the fundamental roots.

(2) Short roots: $\pm\alpha$, $\pm(\alpha+\beta)$

(3) Long roots: $\pm\beta$, $\pm(2\alpha+\beta)$

(4)

$$U_{s_\alpha} = \{x_\alpha(t) \mid t \in \Bbbk\}$$
$$U_{s_\beta} = \{x_\beta(t) \mid t \in \Bbbk\}$$

which are used in the sharp form of the Bruhat decompostion.

The subgroup U consists of elements

$$x_\alpha(t_1)x_\beta(t_2)x_{\alpha+\beta}(t_3)x_{2\alpha+\beta}(t_4) = \begin{pmatrix} 1 & t_1 & t_3+t_1t_2 & t_4+t_1t_3 \\ & 1 & t_2 & t_3 \\ & & 1 & t_1 \\ & & & 1 \end{pmatrix}$$

where $t_i \in \Bbbk$.

We wish to use the sharp form of the Bruhat decomposition to generate G. Define

$$n_\alpha(t) := x_\alpha(t)x_{-\alpha}(t^{-1})x_\alpha(t)$$

$$n_\beta(t) := x_\beta(t)x_{-\beta}(t^{-1})x_\beta(t)$$

for $t \in \Bbbk^*$ then $s_\alpha = n_\alpha(1), s_\beta = n_\beta(1)$, and $h(t,u) = n_\alpha(t)n_\alpha(1)n_\beta(tu)n_\beta(1)$. Via the sharp form of the Bruhat decompostion

$$G = Bn_\alpha U_{s_\alpha} \sqcup Bn_\beta U_{s_\beta}$$

and in particular

$$G = \langle x_r(t) \mid r \in \Phi, t \in \Bbbk \rangle.$$

A theorem of Chevalley guarantees the existence of a bijective morphism of algebraic groups $\theta : G \to G$ such that $\theta \circ F_2 = F_2 \circ \theta$ and $\theta^2 = F_2$, where F_2 is the standard Frobenius map which squares each entry in the matrix.

Let $r = 2^e$, for $e \geq 0$, and $F_r : G \to G$ the standard Frobenius map. Then θ commutes with F_r and setting $F := \theta \circ F_r$, we have that

$$F^2 = \theta^2 \circ F_r^2 = F_2 \circ F_{r^2} = F_{2r^2} = F_{2^{2e+1}}$$

and therefore F is a generalized Frobenius map. The finite group G^F is called a Suzuki group and denoted by ${}^2B_2(q^2)$, where $q = \sqrt{2r}$. These groups are simple except for when $r = 1$.