Frobenius Map & Rational Structures

Suppose that $\Bbbk = \overline{\mathbb{F}}_p$, p a prime. Given $a \geq 1$, there is a unique subfield $\mathbb{F}_q$, $q = p^e$. There is the standard Frobenius map

$$F_q : \Bbbk^n \to \Bbbk^n \text{ defined by } (x_1, \ldots, x_n) \mapsto (x_1^q, \ldots, x_n^q).$$

This map is $\mathbb{F}_q$-linear and a dominant bijective regular morphism.

Let $B \subseteq \Bbbk^n$ be a closed subset, and assume that $V = \mathcal{V}(S)$ for $S \subseteq \Bbbk[x_1, \ldots, x_n]$. Then $F_q(V) \subseteq V$ and consider the fixed points

$$V^{F_q} := \{v \in V \mid F_1(v) = V\} = V \cap \mathbb{F}_q^n$$

is a finite subset of V.

Since F_q is a regular morphism, we can consider the algebra homomorphism

$$F_q^* : \Bbbk[x_1, \ldots, x_n] \to \Bbbk[x_1, \ldots, x_n]$$

which is injective because F_q is bijective. Furthermore, $F_q^*(x_i) = x_i^q$ so $\mathrm{Im}F_q^* = (k[x_1, \ldots, x_n])^q$. Given any $f \in \Bbbk[x_1, \ldots, x_n]$ its coefficients all lie in $\mathbb{F}_{q^e} \subseteq \Bbbk$ and $(F_q^*)^m(f) = f^{q^m}$.

Frobenius Maps.

Definition 1. Let X be an affine variety over $\Bbbk = \overline{\mathbb{F}}_p$ and $F : X \to X$ a morphism. Assume that there is a $q = p^e$ such that the following hold for the pullback $F^* : A \to A$

(a) F^* is injective and $F^*(A) = A^q$.

(b) For each $f \in A$ there exists some $m \geq 1$ such that $(F^*)^m(f) = f^{q^m}$.

When such conditions hold, we say that (X, A) admits an $\mathbb{F}_q$-rational structure, and that F is the Frobenius map for X. Furthermore,

$$X^F := \{x \in X \mid F(x) = x\}$$

which are the $\mathbb{F}_q$-rational points in X.

Proposition 1. *Assume that X is an affine variety over $\mathbb{F}_q$, with Frobenius map $F : X \to X$. Then there exists $n \geq 1$ and a closed embedding $i : X \to \Bbbk^n$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{i} & \Bbbk^n \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F_q} \\ X & \xrightarrow{i} & \Bbbk^n \end{array}$$

Moreover, X^F is finite and F is bijective.

Corollary 1. *Let X be an affine vairety defined over $\mathbb{F}_q$, with corresponding Frobenius map $F : X \to X$. Let*

$$A_0 := \{f \in A \mid F^*(f) = f^q\} \subseteq A.$$

Then for any closed $Y \subseteq X$, the following are equaivalent

(a) $F(Y) \subseteq Y$

(b) $F(Y) = Y$

(c) *The ideal $\mathcal{I}(Y)$ is generated by elements in A_0.*

(d) $Y = \mathcal{V}(S)$ *for some* $S \subseteq A_0$.

If these conditions hold, then $F|_Y : Y \to Y$ is a Frobenius map with respect to $\mathbb{F}_q$.

Algebraic Groups Defined Over $\mathbb{F}_\mathbf{q}$. Let G be an affine algebraic group over $\Bbbk$. We say that G is defined over $\mathbb{F}_q$ if there is a Frobenius map $F : G \to G$ with respect to F_q which commutes with:

$$\mu : G \times G \to G$$
$$(g, h) \mapsto gh$$

and

$$i : G \to G$$
$$g \mapsto g^{-1}$$

That is, F is also a group homomorphism. Consequently,

$$G^F = \{g \in G \mid F(g) = g\}$$

is a finite group. Perhaps more correctly, a *finite algebraic group.*

We now introduce the generalized Frobenius map:

Definition 2. A homomorphism $F : G \to G$ of algebraic groups is called a *generalized Frobenius map* if some power of F is the Frobenius map for an $\mathbb{F}_q$ - rational structure on G.

Theorem 1 (Lang - Steinberg)**.** *Assume that G is a connected affine algebraic group over $\Bbbk$ and $F : G \to G$ is a generalized Frobenius map. Then*

$$L : G \to G$$
$$g \mapsto g^{-1}F(g)$$

is a dominant finite morphism. In particular, L is surjective.

Frobenius Maps & BN-Pairs. Let G be an algebraic group over $\Bbbk$ which has a split BN-pair. If $F : G \to G$ is the Frobenius map on G, we wish to consider B^F and N^F.

Q: Is B^F and N^F a split BN-pair for G^F?

Suppose that $B, N \leq G$ are closed F-stable subgroups which form a *reductive* BN-pair with Weyl group W.

Definition 3. Let G be an affine algebraic group of $\Bbbk$. Suppose G contains two closed subgroups B and N which form a split BN-pair. Write $B = U \cdot H$ where $H = B \cap N$ and $U \lhd B$. We say that G is a group with a *reductive BN-pair* if

(i) The group H is a torus such that $C_G(H) = H$.
(ii) The group U is closed connected and nilpotent.

Q: What does it mean for U to be nilpotent?

A: Define a descending chain of normal subgroups $K_0(G) = G, K_1(G) = [G, G], \ldots, K_{i+1}(G) = [G, K_i(G)]$. We say that G is *nilpotent* if $K_t(G) = \{1_G\}$ for some $t \in \mathbb{N}$.

Definition 4. (1) The *radical* of an algebraic group G is the identity component of the maximal normal solvable subgroup.
(2) The *unipotent radical* of G is defined to be

$$R_u(G) := \bigcap_{\chi \in X(G)} \ker(\chi)$$

(characters here are linear characters $\chi : G \to \mathbb{G}_m$.)

Now that we understand all of the terminology we take $U = R_u(B)$, note that $H = B \cap N$ is a torus, and write $B = U \cdot H$. We wish to show that the Frobenius map satisfies the two properties from the previous lecture:

(BN φ1) $U, N,$ and H are F-stable.
(BN φ2) For any $n \in N$, every coset nH such that $F(nH) \subseteq nH$ contains an F-fixed point.

Seeing that the first property holds is a consequence of the fact that we're assuming that B and N are F-stable, and U is F-stable because U is closed $F(U) \subseteq U$ implies that $F(U) = U$.

To see that the second property holds, let $n \in N$ and suppose that the coset Hn is F-invariant. Then there exists $t \in H$ such that $F(n) = t^{-1}n$. Now $F|_H$ is a generalized Frobenius map and H connected implies that we can apply the Lang-Steinberg theorem. Hence, there exists $h \in H$ such that $t = h^{-1}F(h)$. So,

$$F(hn) = F(h)F(n) = htt^{-1}n = hn$$

and therefore hn is an F-fixed point as required.

By our discussion in the previous lecture, we now have that B^F and N^F form a BN-pair for G^F.

Finite Classical Groups. Let $G \subseteq \mathrm{GL}_n(\Bbbk)$ be one of the following classical groups

$$\begin{cases} \mathrm{GL}_n(\Bbbk) & \text{any } n, \text{any characteristic.} \\ \mathrm{SO}_{2m+1}(\Bbbk) & n = 2m+1, \mathrm{char}(\Bbbk) \neq 2 \\ \mathrm{Sp}_{2n}(\Bbbk) & n = 2m, \text{ any characteristic} \\ \mathrm{SO}^+_{2n}(\Bbbk) & n = 2m, \text{ any characteristic} \end{cases}$$

These groups all have a split BN-pair where

- $B = G \cap B_n(\Bbbk)$
- $N = G \cap N_n(\Bbbk)$
- $U = G \cap U_n(\Bbbk)$

and we can write $B = U \cdot T$, where T is a torus. This BN-pair is reductive. Now consider the Frobenius map

$$F : G \to G$$
$$(a_{ij}) \mapsto (a_{ij}^q)$$

and one can show that B, N are F-stable. From the discussion in the other lecture, B^F and N^F are a split BN-pair for G^F. Hence, the classical finite groups all have a split BN-pair.