Groups with a Split BN-Pair

We want to understand the structure of algebraic groups over a field $k = \overline{\mathbb{F}}_p$. An extremely useful idea has been that of a Tits system. The subgroup B in a BN-pair is a *Borel* subgroup. That is, a subgroup of G which is maximal, connected, closed, and solvable.

BN-Pairs.

Definition 1 (Tits)**.** Let G be a group. We say that G is a group with a *BN-pair* or *admits a Tits system* if there are $B, N \leq G$ such that:

(BN 1) G is generated by B and N; $G = \langle B, N \rangle$.

(BN 2) $H := B \cap N \lhd N$ and $W = N/H$ is generated by a set S of involutions.

(BN 3) Suppose that $s \in S$ and $n_s \in N$ a representative of s. Then $n_s B n_s \neq B$.

(BN 4) $n_s B n \subseteq B n_s n B \cup B n B$ for any $s \in S$ and $n \in N$.

(BN 5) $\bigcap_{n \in N} n B n^{-1} = N \cap B = H$.

Remark 1. (a) BN 5 is referred to as a saturation condition.

(b) W is the Weyl group associated to G, and (W, S) is a Coxeter system.

Let k be a field and $G = \mathrm{GL}_n(k)$. We set

$$B_n(k) := \text{ subgroup of all upper triangular matrices in } \mathrm{GL}_n(k)$$

$$N_n(k) := \text{ subgroup of all monomial matrices in } \mathrm{GL}_n(k)$$

Definition 2. A matrix $M \in \mathrm{Mat}_{n \times n}(k)$ is said to be *monomial* if it has exactly one nonzero entry in each row and each column.

- Note that
$$T_n(k) = B_n(k) \cap N_n(k)$$
is the set of all diagonal matrices in $\mathrm{GL}_n(k)$.
- Define
$$U_n(k) := \left\{ A \in \mathrm{GL}_n(k) \mid \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \right\}$$
These are the matrices with $a_{ii} = 1$ and $a_{ij} = 0$ for $i > j$.

Here are some facts that we wish to use in our discussion:

(a) $B_n(k) = \langle T_n(k), U_n(k) \rangle$

(b) $T_n(k) \cap U_n(k) = \{I_n\}$

(c) $U_n(k) \lhd B_n(k)$

(d) The three facts above imply that $B_n(k) = U_n(k) \rtimes T_n(k)$

(e) $W = N_n(k)/T_n(k) \cong S_n$.

The argument goes like this: each element of $N_n(k)$ has nonzero entries in exactly one row and one column. Applying a permutation matrix, we obtain an element of $T_n(k)$. The group W is generated by a set of involutions and S_n is generated by two cycles, and threfore one identifies $\sigma = (ii{+}1) \in S_n$ with an $s_i \in W$.

Proposition 1. *In the setup we've established, the subgroups $B_n(k)$ and $N_n(k)$ form a BN-pair for $GL_n(k)$ and has Weyl group $W \cong S_n$.*

Proof. We've already talked ourselves through the last statement. We will show that the five axioms $BN1$ through $BN5$ hold.

<u>BN 1:</u> We need to show that $G = \langle B, N\rangle$. Let $g \in G$. Choose $b \in B$ to maximize the total number of zeros at the beginning of each row in bg. The number of zeros that begins each row must be different else one could obtain another zero by row operations. That is we should have chosen b differently. Hence, after an appropriate permutation, we can obtain an element of $B_n(k)$. So, write $n_w bg = b'$ and obtain $g = b^{-1}n_w^{-1}b'$. That we're done is a consequence of the Bruhat decomposition.

<u>BN 2:</u> We need to show that $T_n \lhd N$. This is from our discussion above.

<u>BN 3:</u> We need to show that no n_s normalizes B. We can write $B = X_i \cdot V_i \cdot T_n(k)$ where

$$V_i := \{A \in U_n(k) \mid A = (a_{ij}), a_{i,i+1} = 0\}$$

and

$$X_i := \{I_n + \lambda E_{i,i+1} \mid \lambda \in k\}$$

Well, we can write

$$U_n(k) = X_i \cdot V_i = V_i \cdot X_i$$

and $V_i \cap X_i = \{1\}$ for $1 \leq i \leq n-1$, and since $B_n(k) = U_n(k)T_n(k) = X_iV_iT_n(k)$ as claimed. We now have that

$$\begin{aligned} n_iBn_i &= n_iX_in_i^{-1}n_iV_in_i^{-1}T_n(k) \\ &= X_{-i}V_iT_n \nsubseteq B \end{aligned}$$

<u>BN 4:</u> We need to show that $n_sB_n \subseteq Bn_s \cdot nB \cup BnB$

<u>BN 5:</u> One needs to show that $\bigcap n \in NnBn^{-1} = T_n$. ∎

Split BN-pairs.

Definition 3. Let G be a group with a BN-pair. We say that this is a *split BN-pair* if there exists a normal subgroup $U \lhd B$ such that

(1) $B = UH$ and $U \cap H = \{1\}$. That is, $B = U \rtimes H$.
(2) For any $n \in N$, $n^{-1}Un \cap B \subseteq U$.

We now want to look at groups with a split BN-pair and connections to automorphisms. Let G be a group with a split BN-pair and a finite Weyl group W. Suppose that $\varphi : G \to G$ is an isomorphism satisfying

(BN φ 1) $\varphi(U) = U$, $\varphi(H) = H$, $\varphi(N) = N$ where $B = UH$ and $H = B \cap N$.
(BN φ 2) For $n \in N$, every coset nH such that $\varphi(nH) \subset nH$ contains and element which is fixed by φ.

The fixed point sets $G^\varphi, B^\varphi,$ and N^φ are all subgroups of G. In fact, the pair B^φ, N^φ form a BN-pair for G^φ.

Example 1. We've seen that $\mathrm{GL}_n(k)$ has a BN-pair, where B_n is the standard Borel subgroup of upper triangular matrices and N_n is the subgroup of monomial matrices. Furthermore, if we take $U_n(k)$ to be the unitary group we can write $B = UH$, $H = B \cap N$. Hence $\mathrm{GL}_n(k)$ has a split BN-pair.