## 1. Almost Geometric Quotients

**Proposition 1.** *Let $\pi : X \to X//G$ be a categorical quotient. Then the following are equaivalent:*

(1) *$X$ has a $G$-invariant Zariski dense open subset $U_0$ such that $G \cdot x$ is closed in $X$, for every $x \in U_0$.*

(2) *$X//G$ has a Zariski dense open subset $U$ such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a geometric quotient.*

**Definition 1.** A categorical quotient is said to be *almost geometric* if it satisfies the equivalent conditions above.

**Goal:** Let $X(\Delta)$ be a toric variety of a fan $\Delta$. Our goal is to construct an almost geometric quotient:

$$X(\Delta) \cong (\mathbb{C}^r \setminus Z)//G$$

where

- $Z$ - exceptional set (plays the role of the irrelevant ideal when working with $\mathbb{P}^n$)
- $G$ - linearly reductive group

## 2. Quotient Constructions of Toric Varieties

Let $X(\Delta)$ be a toric variety with no torus factors. That is $N_{\mathbb{R}}$ is spanned by the minimal generators $U_\rho$, where $\rho$ is a ray. The sequence

$$0 \longrightarrow M \longrightarrow \bigoplus_\rho \mathbb{Z}D_\rho \longrightarrow \mathrm{Cl}(X) \longrightarrow 0$$

is exact. We've seen that this is right exact. The map

$$M \longrightarrow \bigoplus_\rho \mathbb{Z}D_\rho$$

$$m \longmapsto \mathrm{div}(\chi^m) = \sum_\rho \langle m, u_\rho \rangle D_\rho$$

If $\mathrm{div}(\chi^m) = 0$, then $\langle m, u_\rho \rangle = 0$, for every ray $\rho$. The $u_\rho$ span $N_{\mathbb{R}}$ by assumptions so $m = 0$.

The functor $\mathrm{Hom}_{\mathbb{Z}}(\ \ ,\mathbb{C}^*)$ is left exact and $\mathbb{C}^*$ is divsible, which gives rise to the following diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X),\mathbb{C}^*) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus \mathbb{Z}D_\rho,\mathbb{C}^*\right) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{C}^*) & \longrightarrow & 1 \\
 & & \| & & \downarrow \wr & & \downarrow \wr & & \\
1 & \longrightarrow & G & \longrightarrow & (\mathbb{C}^*)^{\Delta(1)} & \longrightarrow & T_n & \longrightarrow & 1
\end{array}$$

**Lemma 1** (Structure of $G$)**.**

*Let $G \subset (\mathbb{C}^*)^{\Delta(1)}$ defined as above. Then*

(1) $\mathrm{Cl}(X)$ *is the character group of $G$.*

(2) $G^0$ *is a torus and $G = (\mathbb{C}^r) \times H$, where $H$ is a finite group.*

(3) *Given a basis $e_1, \ldots e_n$ of $M$,*

$$G = \left\{ (t_\rho) \in (\mathbb{C}^*)^{\Delta(1)} \mid \prod_\rho t_\rho^{\langle m, u_\rho \rangle} = 1 \quad \forall m \in M \right\}$$
$$= \left\{ (t_\rho) \in (\mathbb{C}^*)^{\Delta(1)} \mid \prod_\rho t_\rho^{e_i, u_\rho} = 1, 1 \le i \le n \right\}$$

*Proof.* **(b)** The class group is a finitely generated abelian group so $\mathrm{Cl}(X) \cong \mathbb{Z}^l \times H$, where $H$ is a finite group. Then

$$G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X),\mathbb{C}^*) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^l \times H, \mathbb{C}^*) \cong (\mathbb{C}^*)^l \times \mathrm{Hom}_{\mathbb{Z}}(H,\mathbb{C}^*)$$

and we're done since $\mathrm{Hom}_{\mathbb{Z}}(H,\mathbb{C}^*)$ is a finite group.

**(a)** We need to assoaciate $\alpha \in \mathrm{Cl}(X)$ with a character $G \to \mathbb{C}^*$. Well, $g \in G$ is some $\mathrm{Cl}(X) \to \mathbb{C}^*$. So defined

$$G \xrightarrow{\phi_\alpha} \mathbb{C}^*$$

$$g \longmapsto g(\alpha).$$

□

*Remark* 1. $G$ is linearly reductive.

**Example 1.** The fan for $\mathbb{P}^1 \times \mathbb{P}^1$ is $u_1 = e_1, u_2 = -e_2, u_3 = e_2, u_4 = -e_2$. Then $(t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4$ if and only if

$$t_1^{\langle m,e_1\rangle} t_2^{\langle m,-e_1\rangle} t_3^{\langle m,e_2\rangle} t_4^{\langle m,-e_2\rangle} = 1$$

for all $m \in M = \mathbb{Z}^2$. Let $m = (e_1, e_2)$ and we get

$$\begin{aligned} t_1 t_2^{-1} t_3 t_4^{-1} = 1 \Rightarrow t_1 t_2^{-1} = t_3 t_4^{-1} = 1 \\ \Rightarrow G = \{(\mu, \mu, \lambda, \lambda) \mid \mu, \lambda \in \mathbb{C}^*\} \\ \Rightarrow G \cong (\mathbb{C}^*)^2. \end{aligned}$$

## 3. The Exceptional Set

We have a group $G$ and an affine space $\mathbb{C}^{\Delta(1)}$ and we now wish to understand the exceptional set $Z \subseteq \mathbb{C}^{\Delta(1)}$. The two objects that we have $\mathbb{C}^{\Delta(1)}$ and $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X), \mathbb{C}^*)$ depend on $\Delta(a)$. We need $Z$ to encode the rest of the fan.

**Definition 2.** Introduce a variable $x_\rho$, for each $\rho \in \Delta(1)$. The *total coordiante ring* of $X(\Delta)$ is $S = \mathbb{C}[x_\rho \mid \rho \in \Delta(1)]$.

*Remark* 2. Certainly $\mathrm{Spec}\, S$ is $\mathbb{C}^{\Delta(1)}$.

3.1. **Construction of $Z$.** Let $\sigma \in \Delta$ be a cone and define a monomial $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$ the product of those variable corresponding to variables whose rays are not in the cone. To get our exceptional set, we need an ideal:

$$B(\Delta) = \langle x^{\hat{\sigma}} \mid \sigma \in \Delta \rangle \subseteq S$$

is the irrelevant ideal.

Notice that, $\tau \leq \sigma$ implies that $x^{\hat{\tau}}$ is a multiple of $x^{\hat{\sigma}}$ so we can construct $B(\Delta)$ from the maximal cones of $\Delta$. Further, once we have $\Delta(1)$, $B(\Delta)$ determines $\Delta$ uniquely. Then our exceptional set is the variety:

$$Z(\Delta) = \mathbf{V}(B(\Delta)) \subseteq \mathbb{C}^{\Delta(1)}.$$

**Example 2.** Come back to the fan for $\mathbb{P}^1 \times \mathbb{P}^1$ determined by $\Delta(1) = \{u_1, u_2, u_3, u_4\}$, where $u_1 = e_1, u_2 = -e_1, u_3 = e_2, u_4 = -e_2$. For each $u_i$, we get a variable $x_i$, we can compute $Z(\Delta)$

(1) The maximal cones $\text{Cone}(u_1, u_3)$ gives $x_2x_4 = x^{\hat{\sigma}}$. The others are $x_1x_4, x_1x_3, x_3x_2$. Then

$$B(\Delta) = \langle x_2x_4, x_1x_4, x_1x_3, x_3x_2\rangle$$

$$Z = \{0\} \times \mathbb{C}^2 \cup \mathbb{C}^2 \times \{0\}$$

## 4. The Quotient Construction

We now have the following setup

- $C^{\Delta(1)}$ - affine space
- $\text{Hom}_{\mathbb{Z}}(\text{Cl}(X), \mathbb{C}^*)$ - Group
- $B = \langle x^{\hat{\sigma}} \mid \sigma \in \Delta\rangle$ - $V(B)$ is the exceptional set.

We first construct a toric morphism $\mathbb{C}^{\Delta(1)} \setminus Z(B) \to X(\Delta)$. Let $\{e_\rho \mid \rho \in \Delta(1)\}$ be the standard basis for $\mathbb{Z}^{\Delta(1)}$. For each $\sigma \in \Delta$ define

$$\tilde{\sigma} = \text{Cone}(e_\rho \mid \rho \in \sigma(1)).$$

These cones and their faces form a fan $\tilde{Delta} = \{\tau \mid \tau \leq \tilde{\sigma}, \text{ for some } \sigma \in \Delta\}$.

**Proposition 2.**

(1) *$\mathbb{C}^{\Delta(1)} \setminus Z(B)$ is the toric variety of the fan $\tilde{\Delta}$.*

(2) *The map*

$$\mathbb{Z}^{\Delta(1)} \to N$$

$$e_\rho \mapsto u_\rho$$

*of lattices is compatible with fans.*

(3) *The resulting toric morphism $\pi : \mathbb{C}^{\Delta(1)} \setminus Z(B) \to X(\Delta)$ is constant on $G$-orbits.*

*Proof.* **(1)** Let $\tilde{\Delta_0}$ be the fan consisting of $\mathrm{Cone}(e_\rho \mid \rho \in \Delta(1))$ and its faces. Note that $\tilde{\Delta}$ is a subfan of $\tilde{\Delta_0}$. Well, $\tilde{\Delta_0}$ is the fan of $\mathbb{C}^{\Delta(1)}$, and we get the toric variet of $\tilde{Delta}$ by taking $\mathbb{C}^{\Delta(1)}$ and removing the orbits corresponding to cones in $\tilde{\Delta_0} \setminus \tilde{\Delta}$.

The orbit-cone correspondence says this is equivalent to removng the orbit closures of $\tilde{\Delta_0} \setminus \tilde{\Delta}$:

$$\mathbf{V}(x_\rho \mid \rho \in \mathbb{C}) \quad C \text{ is a primitive collection.}$$

So, we remove

$$Z(B) = \bigcup_C \mathbf{V}(x_\rho \mid \rho \in C).$$

**Definition 3.** A subset $C \subset \Delta(1)$ is a primitive collection if

(1) $C \nsubseteq \sigma(1)$, for all $\sigma \in \Delta$.

(2) For every $C' \overset{\nsubseteq}{\text{proper}} C$, there exists $\sigma \in \Delta$ such that $C' \subseteq \sigma(1)$.

**(2)** Define

$$\bar{\pi} : \mathbb{Z}^{\Delta(1)} \to N$$

$$e_\rho \mapsto u_\rho$$

the $u_\rho$ are minimal generators. Notice that $\bar{\pi}_{\mathbb{R}}(\tilde{\sigma}) = \sigma$ by the definition of $\tilde{\sigma}$. Hence, $\bar{pi}$ is compatible with respect to the fans $\tilde{\Delta}, \Delta$.

**(3)** This map $\bar{pi}$ indcues a map of toric

$$(\mathbb{C}^*)^{\Delta(1)} \longrightarrow T_N.$$

Let $g \in G \subseteq (\mathbb{C}^*)^{\Delta(1)}$, by the exact sequence, and $x \in \mathbb{C}^{\Delta(1)} \setminus Z(B)$. One has that

$$\pi(g \cdot x) = \pi(g)\pi(x) = \pi(x)$$

where the first equality is given by the fact that the map $\pi$ is $G$-equivariant, and the second equality is from the fact that $G = \mathrm{Ker}[(\mathbb{C}^*)^{\Delta(1)} \to T_N]$. $\square$

**Theorem 1.** *Let* $X(\Delta)$ *be a toric variety without torus factors and consider the toric morphism*

$$\pi : \mathbb{C}^{\Delta(1)} \setminus Z(B) \longrightarrow X(\Delta).$$

*Then*

(1) $\pi$ *is an almost geometric quotient for the action of* $G$ *on* $\mathbb{C}^{\Delta(1)} \setminus Z(\Delta)$ *so that*

$$X(\Delta) \cong \left(\mathbb{C}^{\Delta(1)} \setminus Z(B)\right) //G.$$

(2) $\pi$ *is a geometric quotient if and only if* $\Delta$ *is simplicial.*