1. Quotients in Algebraic Geometry

While quotients show up in algebraic geometry, at large, we discuss their appearance in toric geometry. Toric varieties have always been viewed as a test-bed, of sorts, for algebraic geoemeters. We look toward the desire to realize a toric variety as an *almost geometric quotient*. We start with two examples that demonstrate what quotients are. The first example gives us what is hoped for, and the second demonstrates the reason we can't get what we want.

Desire. Let G be a linear algebraic group and X an affine G-variety. The action of G allows us to consider two objects:

(1) X/G - the orbits of G. One would hope that each orbit corresponds to a point. That is, orbits are closed.

(2) $k[X]^G$ - the ring of invariants

$$\{f \in k[X] \mid g \cdot f = f \quad \forall g \in G\}.$$

Here's the wish: We desire an embedding

$$k[X]^G \hookrightarrow k[X]$$

which induces a surjective morphism

$$\operatorname{Spec} k[X] \twoheadrightarrow \operatorname{Spec} k[X]^G = X/G.$$

We proceed to consider the following two examples. As we've alluded to, our wish is unfortunately going to be impossible to obtain.

Example 1. Let $G = \mu_2 = \{\pm 1\}$ act on $\mathbb{C}^2$, by $g \cdot (a, b) = (ga, gb)$. We now consider the orbits of this action:

$$\mathcal{O}_{(a,b)} = \{(a, b), (-a, -b)\}$$

$$\mathcal{O}_{(0,0)} = \{(0, 0)\} \text{ - the unique fixed point of the action}$$

The ring of invariants $k[x, y]^{\mu_2} = k[x^2, xy, y^2]$, which is the coordinate ring of $\mathbf{V}(xz - y^2)$, which we've seen is a toric variety, where

$$\mathcal{O}_{(a,b)} \mapsto (a^2, ab, b^2)$$

and yields the map

$$\Phi : \mathbb{C}^2/\mu_2 \to \mathbf{V}(xz - y^2).$$

This gives us the notion that we can make $\mathbb{C}/\mu_2$ into a variety.

It turns out that life isn't quite as nice as we hoped.

Example 2. Let $\mathbb{G}_m$ act diagonally on $\mathbb{C}^2$ by $g \cdot (x, y) = (g \cdot x, g \cdot y)$. If $g \cdot f(x, y) = f(x, y)$, then $f(g \cdot x, g \cdot y) = f(x, y)$, for every $g \in G$. So, $\mathbb{C}[x, y]^{\mathbb{G}_m} = \mathbb{C}$, and $\operatorname{Spec} \mathbb{C}[x, y]^{\mathbb{G}_m} = \{\text{pt.}\}$. Here we have that every orbit is mapped to a single point. So,

$$\pi : \mathbb{C}^2 \to \mathbb{C}^2/\mathbb{G}_m$$

fails to separate orbits.

Remark 1.

(1) Here's a case where we don't get what was desired. Each orbit does not correspond to a unique point in the quotient.

(2) The only closed orbit is $(0, 0) \in \mathbb{C}^2$, the fixed point of the action.

Q: If G acts on an affine variety $\operatorname{Spec} R = X$, is $k[x]^G$ finitely generated?

A: No.

However, if R^G is finitely generated, then $\operatorname{Spec} R^G$ is an affine variety and is our best candidate for the quotient. This is the case when G is linearly reductive.

Definition 1. Let G act on a G-variety X. A *categorical quotient* of X by G is a pair (Y, π) such that Y is a variety, $\pi : X \to Y$ is a morphism

- π is constant on orbits
- for any other variety Z and morphism ψ, which is constant on orbits, ψ factors through π,

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ & \overset{\psi}{\searrow} & \Big\downarrow {\scriptstyle \exists! f} \\ & & Z \end{array}$$

Proposition 1 (Properties of Categorical Quotients)**.** *Let G be a linearly reductive group and X an affine G-variety. Then*

(1) *π is surjective*

(2) *If U is a Zariski open subset of Y, then*

$$\pi^* : \mathcal{O}(U) \to \mathcal{O}(\pi^{-1}(U))^G$$

is an isomorphism.

(3) *If $W \subseteq X$ is Zariski closed and G-stable, then $\pi(W)$ is closed in Y.*

(4) *If $W_1, W_2 \subseteq X$ are Zariski closed, G-stable, and disjoint, then $\pi(W_1) \cap \pi(W_2) = \varnothing$.*

Our definition is fairly abstract, so how do we come up with examples? What are examples of categorical quotients?

If G is a linear algebraic group, and X an affine G-variety, then $\operatorname{Spec} \mathbb{C}[X]^G$ is affine when $\mathbb{C}[X]^G$ is finitely generated. This is always the case when G is linearly reductive.

Remark 2. In characteristic zero, the notions of linearly reductive, reductive, and group theoretically reductive are all equaivalent.

Definition 2. A linear algebraic group G is said to be *linearly reductive* if every rational representation V of G is semisimple. That is V can be written as a direct sum of G-stable simples.

Proposition 2. *If G is a linearly reductive group and X an affine G-variety, then $k[X]^G$ is finitely generated and $\pi : X \twoheadrightarrow \operatorname{Spec} k[X]^G$ is finitely generated and*

$$\pi : X \twoheadrightarrow \operatorname{Spec} k[X]^G$$

corresponds to the inclusion

$$k[X]^G \hookrightarrow k[X].$$

We now characterize the properties of linearly reductive groups. In the following, $\mathcal{R}$ is the Reynold's operator.

Theorem 1 (Characterization of Linearly Reductive)**.**

Let G be a linear algebraic group and V a rational representation. The following are equaivalent:

(1) *G is linearly reductive.*

(2) *For every rational representation V and every $0 \neq v \in V^G$, there exists $f \in (V^*)^G$ such that $f(v) \neq 0$.*

(3) *For every affine G-variety X, there exists a unique G-invariant projection $\mathcal{R} : k[X] \twoheadrightarrow k[X]^G$ such that*

   (a) $\mathcal{R}|_{k[X]^G} = \operatorname{id}|_{k[X]^G}$

   (b) $\mathcal{R}(g \cdot f) = \mathcal{R}(f)$*, for every $f \in k[X]$ and for every $g \in G$.*

Lemma 1. *The Reynold's operator is a $k[X]^G$-module homorophism. That is, for every $f \in k[X]^G$ and $h \in k[X]$, $\mathcal{R}(f \cdot h) = f\mathcal{R}(h)$.*

Theorem 2 (Hilbert's Finiteness Theorem)**.**

If G is linearly reductive and V a rational representation, then $k[V]^G$ is finitely generated over k.

Proof. Let I be the ideal generated by all homogeneous invariants of positive degree. We have the usual grading $k[V] = \bigoplus_{d \in \mathbb{N}} k[V]_d$, which in turn gives a grading for the ring of invariants $k[V]^G = \bigoplus_{d \in \mathbb{N}} k[V]^G_d$. We know that there exists $f_1, \ldots, f_r$ that generate I because $k[V]$ is Noetherian. It is clear that $I \subseteq k[V]^G$ by construction. We wish to show that $k[V]^G = k[f_1, \ldots, f_r]$. We do so by induction on the degree of d.

If $h \in k[V]^G$ of degree zero there is nothing to show. Suppose that h is homogeneous of degree $d \geq 1$. We can write $h = \sum_{i=1}^{r} g_i f_i$, where $g_i \in k[V]$. Without loss of generality, we can assume that the g_i are homogeneous of degree $d - \deg f_i$. Apply the Reynold's operator

$$h = \mathcal{R}(h) = \sum_{i=1}^{r} \mathcal{R}(g_i f_i) = \sum_{i=1}^{r} f_i \mathcal{R}(h_i)$$

and $\mathcal{R}(h_i) \in k[V]^G$ are homogeneous of degree less than d. Hence, by induction $\mathcal{R}(h_i) \in k[f_1, \ldots, f_r]$. We conclude that $h \in k[f_1, \ldots, f_r]$. □

Example 3 (Fulton pg. 33-34)**.**

Let $\sigma \subseteq N_R$ be a strongly convex polyhedral cone and $N' \subseteq N$ a sublattice of finite index. Then the finite group $G = N/N'$ acts on $U_{\sigma,N'}$ such that the induced map on coordinate rings

$$\mathbb{C}[\sigma^\vee \cap M] \xrightarrow{\sim} \mathbb{C}[\sigma^\vee \cap M'] \hookrightarrow \mathbb{C}[\sigma^\vee \cap M'].$$

This inclusion gives a morphism of toric varieties

$$U_{\sigma,N'} \xrightarrow{\pi} U_{\sigma,N}$$

which is a categorical quotient. It is in fact, an example of a geometric quotient.

Definition 3. A *geometric quotient* is a categorical quotient $\pi : X \to X//G$, where the orbits are all closed.

2. Constructing Quotients

We now want to move towards working with more general quotients.

Proposition 3. *Let G act on X and $\pi : X \to Y$ be a morphism that is constant on orbits. If Y has an open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ such that $\pi : \pi^{-1}(V_\alpha) \to V_\alpha$ is a categorical quotient for every α, then π is a categorical quotient.*

Remark 3. The point is we can take a look at what is happening locally.

Example 4. Consider a lattice N and $N' \subseteq N$ of finite index and Δ a fan. We get a toric morphism $\phi : X(\Delta) \to X(\Delta)$, and $G = N/N'$ is the kernel of $T_{N'} \to T_N$. Now, $\phi^{-1}(U_{\sigma,N}) = U_{\sigma',N'}$, for $\sigma \in \Delta$. So the previous proposition gives us that ϕ is a geometric quotient.

3. Good Quotients

We now need to know what to do with arbitrary varieties. Our discussion of quotients for affine varieties will guide us.

Proposition 4 (Dolgachev pg. 93-94)**.**

Let $\pi : X \to Y$ be a G-invariant morphism satisfying

(1) *for any $U \overset{\subseteq}{\text{open}} Y$, $\pi^* : \mathcal{O}(U) \to \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))$ is an isomorphism onto $\mathcal{O}(p^{-1}(U))^G$.*

(2) *If $W \subseteq X$ is a Zariski closed G-invariant subset, then $\pi(W)$ is closed.*

(3) *If W_1, W_2 are two Zariski closed G-invariant disjoint subsets of X, then $\pi(W_1) \cap \pi(W_2) = \varnothing$.*

Then π is a categorical quotient.

Definition 4. A categorical quotient with the properties in the propsition is said to be a *good* categorical quotient.

Corollary 1. *Suppose $\pi : X \to Y$ is a good categorical quotient. Then*

(1) *Two points $x, y \in X$ have the same image in Y if and only if $\overline{Gx} \cap \overline{Gy} \neq \varnothing$.*

(2) *For each $y \in Y$, $\pi^{-1}(y)$ contains a unique closed orbit.*

Proof. (1) Both $\overline{Gx}$ and $\overline{Gy}$ are G-invariant closed subsets of X. If $\overline{Gx} \cap \overline{Gy} \neq \varnothing$, then $\pi(\overline{Gx}) \cap \pi(\overline{Gy}) = \varnothing$, but $\pi(x) = \pi(y)$ so the intersection $\pi(\overline{Gx}) \cap \pi(\overline{Gy})$ cannot be varnothing.

Conversely, if $\pi(x) \neq \pi(y)$ then $\pi^{-1}(x)$ and $\pi^{-1}(y)$ are closed subsets so $\overline{Gx}$ and $\overline{Gy}$ line in different fibers. Hence $\overline{Gx} \cap \overline{Gy} = \varnothing$.

(2) **Uniqueness:** Suppose $\pi^{-1}(y)$ contains two closed orbits Gx and Gz. Then $Gx \cap Gz \neq \varnothing$ but this implies that $\pi(y) = \pi(x) = \pi(z)$ which is ridiculous.

Existence: Requires some facts on the dimension of orbits.

□

Example 5. Let $\mathbb{G}_m$ act on $\mathbb{C}^4$ by $g \cdot (a_1, a_2, a_3, a_4) = (g \cdot a_1, g \cdot a_2, g^{-1} a_3, g^{-1} \cdot a_4)$ and the ring of invariants is

$$k[x_1, x_2, x_3, x_4]^{\mathbb{G}_m} = k[x_1x_3, x_2x_4, x_1x_4, x_2x_3]$$

and

$$\operatorname{Spec} k[x_1, x_2, x_3, x_4]^{\mathbb{G}_m} = \mathbf{V}(xy - zw).$$

We then get a map

$$\Phi : \mathbb{C}^4/\mathbb{G}_m \longrightarrow \mathbf{V}(xy - zw) = \mathbb{C}^4//\mathbb{G}_m$$

$$\mathcal{O}_{(a_1,a_2,a_3,a_4)} \longmapsto (a_1a_3, a_2a_4, a_1a_4, a_2a_3).$$

Here are some properties that we should observe:

(1) Φ is surjective

(2) If $p \in V(xy - zw) - \{0\}$, then $\Phi^{-1}(p)$ is a single closed $\mathbb{G}_m$-orbit.

(3) $\Phi^{-1}(0) = \mathbb{C}^2 \times \{(0,0)\} \cup \{(0,0\} \times \mathbb{C}^2 \supseteq \mathbb{G}_m -$ orbits

That is

$$\{\text{Closed}\mathbb{G}_m\text{-orbits}\} \longleftrightarrow \mathbf{V}(xy - zw)$$

and since every fibre contains a unique closed orbit we have a bijection

$$\{\text{closed} G\text{-orbits in} X\} \longleftrightarrow \{\text{points in} X//G\}.$$