Some Recollections & Setup

Let G be a finite group of order n, and k and algebraically closed field of characteristic p.

- We wish to study the ring of invariants

$$k[x_1, \dots, x_n]^G = \{f \in k[X_1, \dots, x_n] \mid (g \cdot f)(v) = f(v), \quad \forall\, v \in V, \quad \forall\, g \in G\}.$$

- Finiteness Issues:

  (1) Structural:

  Q: Is the ring of invariants Noetherian?

  (2) Combinatorial:

  Q: What is the rate of growth of the sequence of integers

$$\dim_k \left(k[V]_d^G\right) = \dim_k S^d(V^*)^G?$$

  (3) Homological:

  Q: Is the ring of invariants Cohen-Macaulay?

  Q: What can we say about the length of syzygy chains?

- Our representations will all be faithful. Meaning that

$$\rho : G \hookrightarrow \mathrm{GL}(V)$$

  is injective, where V is a finite-dimensional vector space over k.

Semisimple

We begin by setting out some definitions.

Definition 1.

(1) A *simple* representation has no subrepresentations other than itself and the trivial one.

(2) A representation is said to be *semisimple* if it can be written as a direct sum of simples.

(3) A representation is said to be *indecomposable* if it cannot be written as a direct sum of subrepresentations.

Theorem 1 (Maschke's Theorem)**.** *Let G be a finite group and k a field so that* $\operatorname{char} k \nmid \#G$*. Then every representation of G is semisimple. Equivalently, the group algebra kG is semisimple.*

Example 1. Let k be of characteristic p, and C_p the cyclic group of order p, written multiplicatively. If $\rho : C_p \to k^*$ is a representation, then

$$1 = \rho(x^p) = \rho(x)^p \quad \text{for every } x \in C_p.$$

One sees that every element of C_p acts as a pth root of unity. However,

$$(\lambda - 1)^p = x^p - 1 \;\; \forall\, \lambda \in k^*$$

and therefore every pth root of unity is 1. Hence, $\rho : C_p \to k^*$ is the trivial homomorphism.

Q: How many simple representations are there?

Let $\rho : G \to \mathrm{GL}(V)$ be an arbitrary representation.

A:

- Nonmodular Representations

  # of Conjugacy classes = # of iso classes of simples.

- Modular Representations

  Let $\#G = p^n r$, and k have $\operatorname{char} p, (p, r) = 1$.

  # of iso classes of simples = r.

Examples in Modular Representation Theory

Our setup is as follows:

- k is a field of arbitrary characteristic, $p > 0$.
- G is a finite group, $\#G = n$.

One can consider the following cases:

- Case 1: Nonmodular $\#G \in k^*$
  - Strong Nonmodular
  - Weak Nonmodular
- Case 2: Modular $\#G \equiv 0 \pmod{p}$.

The Transfer Homomorphism

A basic tool in much of representation theory of finite group is to "average" with respect to the order of G. We cannot do this when the characteristic of k divides the order of G. We let $H \leq G$ and define the *transfer homomorphism*

$$\mathrm{Tr}_H^G : k[V]^H \longrightarrow k[V]^G$$

$$f \longmapsto \sum_{gH \in G/H} g \cdot f(x) = \sum_{gH \in G/H} f(g^{-1} \cdot x).$$

Remark 1.

(1) We're letting the sum run over the representatives of left cosets of H in G.

(2) For $H \leq G$, we know that $k[V]^G \hookrightarrow k[V]^H$ as a subalgebra.

Moreover, for every $f \in k[V]^G, h \in k[V]^H$ with $\deg h \geq 1$

(1) $\mathrm{Tr}_H^G(f) = [G : H]f$

(2) $\mathrm{Tr}_H^G(f \cdot h) = f\,\mathrm{Tr}_H^G(h)$

which gives us that Tr_H^G is an $k[V]^G$-module homomoprhism. We can consider

$$k[V]^G \hookrightarrow k[V]^H \xrightarrow{\mathrm{Tr}_H^G} k[V]^G$$

which is equivalent to multiplication by $[G : H]$. Hence, when $[G : H] \in k^*$, Tr_H^G is surjective.

Reynold's Operator

When $[G:H]$ is invertible in k,

$$\mathcal{R}_H^G(f) = \frac{1}{[G:H]} \sum_{gH \in G/H} g \cdot f$$

is the Reynold's operator. In this case,

$$k[V]^H = k[V]^G \bigoplus \operatorname{Ker} R_H^G$$

and in particular if h is the trivial subgroup

$$\mathcal{R}^G : k[V] \longrightarrow k[V]^G$$

$$f \longmapsto \frac{1}{\#G} \sum g \in Gg \cdot f$$

is the projection onto the ring of invariants.

Remark 2. If the characteristic of k divides the order of G, then $\mathcal{R}^G$ is never surjective.

Example 2. Let $\mathbb{F}_2$ be the finite field of order 2 and $G = C_2$. Consider the usual representation $\rho : C_2 \to \mathrm{GL}_2(\mathbb{F}_2)$ where

$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\rho(-1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note that xy is invaraint, and $x^2, xy,$ and y^2 is a basis for $\mathbb{F}_2[V]_2$. We now make some computations and an observation:

$$\mathrm{Tr}^G(x^2) = \sum_{g\in G} g\cdot x^2 = x^2+y^2 = \sum_{g\in G} g\cdot y^2 = \mathrm{Tr}^G(y^2)$$

$$\mathrm{Tr}^G(xy) = \sum g\in Gg\cdot xy = xy+xy = 2xy = 0.$$

Hence, every degree two form has no xy term. That is $xy \in \mathbb{F}_2[V]_2^G$, but $xy \notin \operatorname{Im}\mathrm{Tr}^G$.

Theorem 2 (Feshbach-Derksen)**.** *Let $\rho : G \to \mathrm{GL}_n(k)$ be a representation of a finite group G. If $\operatorname{char} k$ divides $\#G$, then Tr^G is not surjective.*

Proof. Let $\operatorname{char} k = p > 0$, a prime. Since $p \mid \#G$, there exists $h \in G$ such that $|h| = p$ by Cauchy's theorem. Define $H \leq G$ to be the cyclic subgroup generated by h. Let $g_1, \ldots, g_k$ be a left transversal of H in G. We know that $V^H \neq \{0\}$ because $\#H = p$. Choose $0 \neq v \in V^H$ and note that for any $f \in k[V]$

$$\begin{aligned}\mathrm{Tr}^G(f)(v) &= \sum_{g\in G}(g\cdot f)(v)\\ &= \sum_{g\in G}(f(g^{-1}\cdot v)\\ &= \sum_{j=1}^{k}\sum_{i=0}^{p-1} f(g_j^{-1}h^{-i}\cdot v)\\ &= \sum_{j=1}^{k} pf(g_j^{-1}\cdot v)\\ &= 0.\end{aligned}$$

Suppose that Tr^G is surjective. Choose a system of parameters $f_1, \ldots, f_n \in k[V]^G$. Then $f_1, \ldots, f_n$ is also a system of parameters in $k[V]$. Let $f \in k[V]$. There exists $F_1, \ldots, F_n$ such that $f = \sum_{i=1}^n f_iF_i$.

[Finish this proof]

□

Cohen-Macaulay

We want to understand a nice property of rings. To be Cohen-Macaulay is to say that the *depth* and Krull dimension are equal. We then will take a look at what we know about when the ring of invariants is Cohen-Macaulay.

Definition 2. Suppose that $R = \bigoplus_{d \in \mathbb{N}_0} R_d$ is a graded k-algebra, as usual $R_0 = k$. A set $f_1, \dots f_r \in R$ of homogeneous elements is said to be a *homogeneous system of parameters* (hsop) if

(1) the f_i are algebraically independent.

(2) $k[f_1, \dots, f_r] \subseteq R$ is module-finite.

Remark 3. Some more definitions:

(1) the $f_1, \dots, f_r \in k[V]^G$ are *primary invariants*

(2) $k[V]^G = Fg_1 + Fg_2 + \cdots + Fg_s$, where $F = k[f_1, \dots, f_r]$, the g_i are said to be the *secondary invariants.*

Fact: The Noether Normalization Theorem guarantees that invariant rings have a hsop.

Definition 3. Let R be a Noetherian graded ring and M a finitely generated R-module.

(1) A sequence $r_1, \dots, r_s \in R$ is said to be *M-regular* if $M/\langle r_1, \dots, r_s \langle M \neq 0$ and r_i is a NZD on $M/\langle r_1, \dots, r_{i-1} \rangle M$.

(2) Let $I \subseteq R$ be an ideal, with $IM \neq M$. Then $\operatorname{depth}_M(I)$ is the maximal length of an M-regular sequence.

(3) The module M is said to be *Cohen-Macaulay* if for all maximal ideals $\mathfrak{m} \in \operatorname{Supp}(M)$,

$$\operatorname{depth}(M_{\mathfrak{m}}) = \dim_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}).$$

Proposition 1. *Let R be a Noetherian graded k-algebra with $R_0 = k$. The following are all equivalent:*

(1) *R is Cohen-Macaulay.*

(2) *every hsop is R-regular.*

(3) *If $f_1, \dots, f_r$ is an hsop, then R is a free-module over $k[f_1, \dots, f_r]$.*

Theorem 3 (Hochster-Eagon)**.** *If* char(k) *does not divide* $\#G$*, then* $k[V]^G$ *is Cohen-Macaulay.*

Example 3. Let $G = \langle\sigma\rangle = C_p$ be the cyclic group of order p written multiplicatively, and char $k = p > 0$. Consider the action of G on $k[x_1, x_2, x_3, y_1, y_2, y_3]$ by

$$\sigma \cdot x_i = x_i \text{ and } \sigma \cdot y_i = x_i + y_i.$$

Remark 4.

(1) The x_i are invariant by definition of the action.

(2) $u_{ij} = x_i y_j - x_j y_i$ are invariant $1 \leq i, j \leq 3$.

One can extend the sequence x_1, x_2, x_3 to a hsop for $k[V]^G$. However,

$$0 = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = u_{23}x_1 - u_{13}x_2 + u_{12}x_3$$

shows that x_1, x_2, x_3 is not $k[V]^G$-regular because $u_{12} = x_1y_2 - x_2y_1 \notin \langle x_1, x_2\rangle k[V]^G$. We conlcude that $k[V]^G$ is not Cohen-Macaulay.