Hardy spaces are function spaces of analytic functions, which are, in some sense, analogs to the Lebesgue spaces one encounters during a first course in measure theory. The study of Hardy spaces is connected to many other areas of mathematics: harmonic analysis, Fourier analysis, singular integrals, control theory, and interpolation problems, just to name a few. Our concern is understanding basic theorems and ideas behind interpolation problems.

Recall that a *Banach Space* is a complete normed space and that a *Hilbert Space* is a complete inner product space. Every Hilbert space is a Banach space. Since there exist norms that do not arise from an inner product, there are Banach spaces which are not Hilbert spaces. To say that a space is *complete* is to say that every Cauchy sequence converges.

1 Interpolation Problems

The basic idea behind an interpolation problem is to find a function $f(z)$ with a certain set of properties such that $f(z_j) = w_j$ for data $z_1, z_2, z_3, \ldots$ and $w_1, w_2, w_2, \ldots$. We discuss a few of these types of problems below.

Interpolation Problem 1 (Lagrange)**.** Given distinct numbers $z_j \in \mathbb{F}$ and another set of arbitrary numbers $w_j \in \mathbb{F}, 1 \leq j \leq m$, find a polynomial, of degree n or less, such that

$$p(z_j) = w_j \quad \text{for all } j.$$

A special situation of this is to produce $l_i(z)$, again of degree n or less, such that

$$l_i(z_j) = w_j = \delta_{ij} \quad \text{for } 1 \leq i, j \leq m.$$

This is an easy problem to solve, and the solution is given by the Lagrange Interpolation polynomials

$$l_i(z) = \frac{\prod\limits_{i \neq j} (z - z_j)}{\prod\limits_{i \neq j} (z_i - z_j)}.$$

The Nevanlinna-Pick Interpolation Problem is classic. It was studied by Nevanlinna and Pick independently in the early 20^{th} century. It is more sophisticated than our previous example. The setting is now the open unit disk $\mathbb{D}$ in the complex plane.

Interpolation Problem 2 (Nevanlinna-Pick)**.** Given $z_1, z_2, \ldots, z_n$ and $w_1, w_2, \ldots, w_m$ in the open unit disk, under what conditions does there exist an $f(z)$ analytic on $\mathbb{D}$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$, such that

$$f(z_j) = w_j \quad \text{for all } 1 \leq j \leq m.$$

One takes another step in complexity when considering the Carleson Interpolation Problem. A number of results came out of the study of this problem including the desire to understand H^∞ as a Banach algebra. It was the original impetus for the definition of Carleson measures and the Corona theorem.

Interpolation Problem 3 (Carleson)**.** Suppose that $\{z_j\}_{j\in\mathbb{N}}$ is a sequence of distinct complex numbers in $\mathbb{D}$, does there exist a bounded analytic function $f(z)$ such that $f(z_j) = w_j$, for each bounded sequence $\{w_j\}_{j\in\mathbb{N}}$.

We can recast this statement in the following way. The space ℓ^∞ is the space of bounded sequences. For $x \in \ell^\infty$,

$$\|x\|_\infty = \sup\{|x_1|, |x_2|, \ldots, |x_n|, \ldots\}$$

one can show that ℓ^∞ is a Banach space. We can define a mapping

$$\begin{aligned} H^\infty &\longrightarrow \ell^\infty \\ f &\longmapsto \{f(z_j)\}_{j\in\mathbb{N}} \end{aligned}$$

Under what conditions is this mapping surjective?

Fact: Any Carleson sequence $\{z_j\}_{j\in\mathbb{N}}$ must satisfy the Blaschke condition

$$\sum_{j\in\mathbb{N}} (1 - |z_j|) < \infty.$$

2 Hardy Spaces

Our setting is the complex plane. One can pursue two theories simultaneously. That which takes place in a half-plane and that which occurs in the unit disk. One moves seamlessly between these theories without producing any of the details. This is possible through a conformal mapping of the half-plane and the unit disk.

For an analytic function in the unit disk, call it f, we say that $f \in H^p$ if

$$\sup_{0<r<1} \int_0^{2\pi} \left|f(re^{i\theta})\right|^p < \infty$$

which defines a norm $\|\cdot\|_p$ on the space. For $p = \infty$,

$$\|f\|_\infty = \sup_{0<r<1} |f(z)|$$
$$H^\infty(\mathbb{D}) = \left\{ f \text{ anlaytic on } \mathbb{D} \;\middle|\; \sup_{\substack{0<r<1 \\ \theta\in\mathbb{R}}} \left|f(re^{i\theta})\right| < \infty \right\}$$

We will be particularly interested in these functions as they are essentially bounded. As one can imagine, such functions have some nice properties.

The definitions for the upper-half-plane are as follows:

$$\|f\|_p^p = \sup_{y>0} \int_{-\infty}^{+\infty} |f(x+iy)|^2 \; dy$$
$$H^p(\mathbb{C}_+) = \left\{ f \text{ analytic on the upper half-plane } \;\middle|\; \|f\|_p < \infty \right\}$$

which is a Hilbert space of analytic function on the upper-half-plane.

Q: What happens with these functions on the real axis?

A: A theorem of Fatou guarantees the existence of boundary values on the real axis.

3 Blaschke Products

We begin with a discussion of infinite products because they will be important to us later. We are familiar with what it means for a series to converge. In analog, what does it mean for an infinite product $\prod_{n=1}^{\infty} p_n$ to converge? We define the partial products $P_N = \prod_{n=1}^{N} p_n$ and say that $\prod_{n=1}^{\infty} p_n = P$ if the sequence $\{p_N\}_{N=1}^{\infty}$ converges to P.

If one of the p_n is zero, then the idea of having an infinite product is trivial. So, suppose that $\{p_n\}_{n\in\mathbb{N}}$ is a sequence of nonzero complex numbers. Notice that the quotient of the partial products $\frac{p_{N+1}}{p_N} = p_n$. Now, $P_N \to P$ implies that $\lim_{n\to\infty} p_n = 1$. Hence, a necessary condition for an infinite product to converge is that the n^{th}-term goes to 1.

In the following discussion, we use the principal branch of the logarithm. Suppose that $\sum_{n=1}^{\infty} \log p_n$ is convergent. Let $S_N = \sum_{n=1}^{N} \log p_n$ be the N-th partial sum. By convergence of $S_N \to S$, then $\exp(S_N) \to \exp(S)$. However,

$$
\begin{aligned}
\exp(S_N) &= \exp\left(\sum_{n=1}^{N} \log p_n\right) \\
&= \prod_{n=1}^{N} \exp(\log p_n) \\
&= \prod_{n=1}^{N} p_n \\
&= P_N
\end{aligned}
$$

and we conclude that $P_N \to \exp(S)$ and $\prod_{n=1}^{\infty} p_n = \exp(S) \neq 0$.

Proposition 1. *Let $\Re z_n > 0$ for all $n \geq 1$. Then $\prod_{n=1}^{N} z_n$ is convergent to a nonzero complex number if and only if $\sum_{n=1}^{\infty} \log z_n$ converges.*

In a similar vein,

Proposition 2. *Let $\Re z_n > -1$. Then the series $\sum_{n=1}^{\infty} \log(1+z_n)$ converges absolutely if and only if $\sum_{n=1}^{\infty} z_n$ converges absolutely.*

These ideas enable us to make the proper definition for an infinite product to be convergent.

Definition 1. If $\Re z_n > 0$ for all n, the infinite product $\prod_{n=1}^{\infty} z_n$ is said to *converge absolutely* if the series $\sum_{n=1}^{\infty} \log z_n$ converges absolutely.

Now that we know what it means for an infinite product to be convergent, we are able to discuss the infinite products that are analytic on the unit disk. If $\{z_j\}_{j\in\mathbb{N}}$ is a sequence of complex numbers in $\mathbb{D}$, which are zeros of an $f \in H^p$, then they satisfy the Blaschke condition $\sum_{j=1}^{\infty}(1-|z_j|) < \infty$.

Theorem 1. *Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence of points in $\mathbb{D}$ which are nonzero and satisfy the Blaschke condition. Then the Blaschke product*

$$B(z) = \prod_{j=1}^{\infty} \frac{-\bar{z}_j}{|z_j|} \frac{z - z_j}{1 - \bar{z}_j z}$$

converges on $\mathbb{D}$.

Proof. Define

$$b_j(z) := \frac{-\bar{z}_j}{|z_j|} \frac{z - z_j}{1 - \bar{z}_j z}$$

to be an elementary Blaschke factor. Notice that

$$b_j(0) = \frac{-\bar{z}_j}{|z_j|} \frac{-z_j}{1} = \frac{|z_j|}{|z_j|} = 1.$$

Now, the modulus of an elementary Blaschke factor is

$$\left|b_j(e^{i\theta})\right| = \frac{\left|e^{i\theta} - \rho e^{i\gamma}\right|}{\left|1 - \rho e^{-i\gamma} e^{i\theta}\right|} = \left|e^{i\theta}\right| \frac{\left|1 - \rho e^{i(\gamma-\theta)}\right|}{\left|1 - \rho e^{i(\theta-\gamma)}\right|} = 1.$$

By the maximum modulus principle, we must have that $|b_j(z)| < 1$ for all $z \in \mathbb{D}$.

Via some algebra we obtain an identity we will need to show a sketch of why the product converges:

$$\begin{aligned}
b_j(z) &= \frac{-\bar{z}_j}{|z_j|} \frac{z_j - z}{1 - \bar{z}_j z} \\
&= \frac{-\bar{z}_j z_j}{|z_j|} \frac{1 - \frac{z}{z_j}}{1 - \bar{z}_j z} \\
&= -\frac{|z_j|^2}{|z_j|} \frac{1 - \frac{z}{z_j}}{1 - \bar{z}_j z} \\
&= -|z_j| \frac{1 - \bar{z}_j z_j + (\bar{z}_j - 1/z_j) z}{1 - \bar{z}_j z} \\
&= -|z_j| \left(1 + \frac{(|z_j|^2 - 1)\frac{z}{z_j}}{1 - \bar{z}_j z}\right) \\
&= -|z_j| - \frac{|z_j|^2 - 1}{1 - \bar{z}_j z} \frac{z\,|z_j|}{z_j}
\end{aligned}$$

which gives us that

$$b_j(z) = -1 + (|z_j| - 1)\left[-1 - \frac{|z_j| + 1}{1 - \bar{z}_j z}\frac{z\,|z_j|}{z_j}\right].$$

The point is that $\prod_{j=1}^{\infty}(b_j(z) + 1 - 1)$ converges absolutely if and only if $\sum_{j=1}^{\infty}|1 - b_j(z)|$ is finite. Well,

$$|1 - b_j(z)| \le (1 - |z_j|) + (1 - |z_j|)\frac{2\,|z|}{1 - |z_j|\,|z|}$$

and one obtains the convergence from the fact that the sequence $\{z_j\}_{j\in\mathbb{N}}$ satisfies the Blaschke condition. $\square$

We note that we now know that $B(z)$ is an example of a function in H^∞ because it is bounded and analytic on the open unit disk. A consequence of the Lebesgue Dominated Covergence Theorem is that the zeros of an $f \in H^\infty$ satisfy the Blaschke condition. Further, from a theorem of F. Riesz, if $f \in H^p$ and not identically zero, then $f(z) = g(z)B(z)$, where $g(z) \neq 0$ on the open unit disk and $g \in H^p$ and $B(z)$ is a Blaschke product. That is, we can use Blaschke products to factor out the zeros.

4 Discussion of Interpolation Problems

We mentioned three interpolation problems in the introduction. We will discuss the last two in this section.

Interpolation Problem 2 (Pick)**.** Suppose that $\{z_1, \ldots, z_m\}$ is a finite set of distinct points in $\mathbb{D}$. For which $\{w_1, \ldots, w_m\}$ does there exist an f, which is analytic on $\mathbb{D}$, such that

$$f(z_j) = w_j \quad \text{for } 1 \leq j \leq m?$$

One can present a solution to this problem via quadratic forms.

Theorem 2 (Pick)**.** *There exists an $f : D \longrightarrow \overline{\mathbb{D}}$, which is analytic, that solves the interpolation problem above if and only if the quadratic form*

$$Q(t_1, \ldots, t_n) = \sum_{j,k=1}^{n} \frac{1 - w_j \bar{w}_k}{1 - z_j \bar{z}_k} t_j \bar{t}_k$$

is nonnegative.

The statement of this theorem is tantamount to stating that the matrix

$$\left(\frac{1 - \bar{w}_j w_k}{1 - \bar{z}_j z_k} \right)_{j,k=1,\ldots,n}$$

is positive semidefinite.

We get a more "adequate" or "best" result if we consider inner functions.

Definition 2. An *inner function* is a function $f \in H^\infty$ such that $\left| f(e^{i\theta}) \right| = 1$ almost everywhere.

We've already encountered an example of an inner function in Blaschke products. Another example of an inner function are *singular integrals*

$$S(z) = \exp\left(- \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu(\theta) \right).$$

The name "inner function" begs the question: Is there such a thing as an outer function? Yes, but they will not be important to our discussion.

Definition 3. A function $f \in H^p$ is an *outer function* if $g \in H^p$ and $\left| g(e^{i\theta}) \right| - \left| f(e^{i\theta}) \right|$ almost everywhere implies that $|g(z)| \leq |f(z)|$ for all $z \in \mathbb{D}$.

We now discuss why inner fuctions are of interest to our situation. Here's the setup for the following theorem. Let $\{z_j\}_{j \in \mathbb{N}}$ be a sequence of distinct points in $\mathbb{D}$ and $\{w_j\}_{j \in \mathbb{N}}$ any sequence of complex numbers.

Supposing there is some $f \in H^\infty$ that solves the interpolation problem, there is more than one solution in H^∞, when the z_j satisfy the Blaschke condition $\sum\limits_{j\in\mathbb{N}}(1-z_j) < \infty$.

In the situation where the sequences are finite, we can solve the interpolation problem with some multiple of a Blaschke product. Here's a corollary to Pick's theorem.

Corollary 1. *Suppose the quadratic form Q_m is positive semidefinite. Then the interpolating function is a multiple of a Blaschke product of degree at most m.*

Let

$$B(z) = \prod_{j=0}^{\infty} \frac{z - z_j}{1 - \bar{z}_j z}$$

be the Blaschke product with zeros $z_1, \ldots, z_m$. Suppose there is a solution $f_0 \in H^\infty$ which does the interpolation. The minimal norm of the function which do the interpolation is given by

$$\inf_{g \in H^\infty} \|f_0(z) - B(z)g(z)\|_\infty = \inf_{g \in H^\infty} \left\|\overline{B(z)}f_0(z) - g(z)\right\|_\infty$$

Since $\overline{B(z)}f(z)$ is continuous on the circle, there exists a unique interpolating function $f \in H^\infty$ of minimal norm.

Can we get a result like this in the case when the sequences are infinite? This is where our interest in inner function comes into play in a serious way.

Theorem 3 (Nevanlinna)**.** *If there are two distinct functions of unit norm in H^∞ that do the interpolation, then there is an inner function that does the interpolation.*

We now come to the Carleson interpolation problem. We begin with a definition.

Definition 4. A sequence $\{z_j\}_{j\in\mathbb{N}}$ in the open unit disk is said to be an *interpolating sequence* if every interpolation problem

$$f(z_j) = w_j \quad \text{for } j \in \mathbb{N}$$

with $\{w_j\}_{j\in\mathbb{N}}$ bounded has a solution in H^∞.

It was with the desire to understand H^∞ as a Banach algebra that interpolating sequences were first considered. We restate Carleson's interpolation problem in the upper half plane.

Interpolation Problem 3 (Carleson)**.** Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in the upper half plane. Determine when every interpolation problem

$$f(z_j) = w_j \quad \text{for } j \in \mathbb{N}$$

with the sequence of w_j bounded has a solution $f \in H^\infty$.

Notice that if $\{z_j\}_{j\in\mathbb{N}}$ is an interpolating sequence, the there is a solution in H^∞ for every $\{a_j\}$ in ℓ^∞. Consider,

$$N = \{f \in H^\infty \mid f(z_j) = 0\,\forall j \in \mathbb{N}\}\,.$$

We are uninterested in these functions. Define

$$\begin{aligned} T : H^\infty/N &\longrightarrow \ell^\infty \\ f &\longmapsto \{f(z_j)\}_{j\in\mathbb{N}}. \end{aligned}$$

It turns out that this is a surjective bounded linear operator when $\{z_j\}_{j\in\mathbb{N}}$ is an interpolating sequence.

The following theorem solves the problem, as far as we're concerned.

Theorem 4. *If $\{z_j\}_{j\in\mathbb{N}}$ is a sequence in the upper half plane, then the following are equivalent:*

1. *The sequence is an interpolating sequence.*
2. *There is $\delta > 0$ such that*

$$\prod_{j,j\neq k} \left| \frac{z_k - z_j}{z_k - \bar{z}_j} \right| \geq \delta \quad \textit{for } k \in \mathbb{N}.$$

5 What's Going on for Rational Functions?

In Fuhrmann's book, we concerned ourselves with the space

$$\mathbb{F}((z)) = \mathbb{F}[z] \oplus z^{-1}\mathbb{F}[[z^{-1}]].$$

We start with a couple of definitions.

Definition 5.

1. A complex polynomial $q(z)$ is called *stable* if all its zeros lie in the open upper half-plane.
2. A complex polynomial is called *antistable* if all its zeros lie in the closed lower half-plane.

Here are the definitions of the function spaces that Fuhrmann is working with in his book.

$$RL^\infty = \left\{ \frac{p(z)}{q(z)} \;\middle|\; p(z), q(z) \in \mathbb{C}[z], \deg p \leq \deg q, q(\zeta) \neq 0, \zeta \in i\mathbb{R} \right\}$$

$$RH_+^\infty = \left\{ \frac{p(z)}{q(z)} \in RL^\infty \;\middle|\; q(z) \text{ stable } \right\}$$

$$RH_-^\infty = \left\{ \frac{p(z)}{q(z)} \in RL^\infty \;\middle|\; q(z) \text{ antistable } \right\}$$

$$RL^2 = \left\{ \frac{p(z)}{q(z)} \in RL^\infty \;\middle|\; \deg p < \deg q \right\}$$

$$RH_+^2 = \left\{ \frac{p(z)}{q(z)} \in RL^2 \;\middle|\; q(z) \text{ stable } \right\}$$

$$RH_-^2 = \left\{ \frac{p(z)}{q(z)} \in RL^2 \;\middle|\; q(z) \text{ antistable } \right\}$$

We make a note that RL^2 is an inner product space. The inner product is given by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(it)\overline{g(it)}\, dt$$

for $f, g \in RL^2$. It is a fact from complex analysis that such integrals can be evaluated by using the residue calculus.

5.1 Intertwining Maps and Interpolation

We begin with a definition of what it means for a map to intertwine a bounded analytic function in this context. Let $m(z) \in RH_+^\infty$ be an inner funtion and define

$$H(m) := RH_+^2 / mRH_+^2.$$

This quotient makes sense because mRH_+^2 is a RH_+^∞-submodule of RH_+^2.

Definition 6. We say that a map $X : H(m) \longrightarrow H(m)$ is an *intertwining map* if for all $\Psi(z) \in RH_+^\infty$, we have that

$$XT_\Psi = T_\Psi X.$$

Recall that intertwining maps are RH_+^∞-module homomorphisms in $H(m)$.

Interpolation Problem 4. (for RH_+^∞)

1. Given $w_j \in \mathbb{C}$, and $z_j \in \mathbb{C}_+$, $1 \leq j \leq m$, find $\Psi(z) \in RH_+^\infty$ such that

$$\Psi(z_j) = w_j \quad \text{for } i = 1, 2, \ldots, m.$$

2. Given $w_j \in \mathbb{C}$ and $z_j \in \mathbb{C}_+$, $1 \leq j \leq m$, find $\Psi_i(z) \in RH_+^\infty$ for which

$$\Psi_i(z) = \begin{cases} 0 & j \neq i \\ w_j & j = i \end{cases}$$

The statement in (2) is similar to the Lagrange Interpolation Problem that we discussed in the beginning.

Proposition 3. *Given $w_j \in \mathbb{C}$ and $z_j \in \mathbb{C}_+$, $1 \le j \le m$, define*

$$m(z) := \prod_{j=1}^{m} \frac{z - z_j}{z + \bar{z}_j}$$
$$\pi_{z_j} := \prod_{j \neq i} \frac{z - z_j}{z + \bar{z}_j}$$
$$m_{z_j} := \frac{z - z_j}{z + \bar{z}_j}$$
$$l_{z_j} := \frac{1}{z + \bar{z}_j}$$

Then

1. *There exist solutions $\Psi_j(z) \in RH_+^\infty$ given by*

$$\Psi_j(z) = w_j \pi_{z_j}(z_j)^{-1} \pi_{z_j}(z)$$

2. *A solution of the interpolation problem is given by*

$$\Psi(z) = \sum_{j=1}^{m} w_j \pi_{z_j}(z_j)^{-1} \pi_{z_j}(z).$$

Proposition 4. *Given an inner function $m(z) \in RH_+^\infty$ having distinct zeros z_j, $1 \le j \le m$, then*

1. *We have factorizations*

$$m(z) = \pi_{z_j} z) m_{z_j}(z).$$

2. *The space $H(m_{z_j})$ is spanned by the functions $l_{z_j}(z)$, and we have*

$$RH_+^2 = H(m_{z_j}) \oplus m_{z_j} RH_+^2$$

3. *The set $\{\pi_{z_j}(z) l_{z_j}(z)\}_{i=1}^m$ forms a basis for $H(m)$.*

4. *A map $X : H(m) \longrightarrow H(m)$ is an intertwining map if and only if there exists $\Psi(z) \in RH_+^\infty$ such that $X = T_\Psi$.*

5. *If $\Psi(z) \in RH_+^\infty$, we have, for $T_\Psi : H(m) \longrightarrow H(m)$*

$$\|T_\Psi\| \le \|\Psi\|_\infty .$$

These statements give an explicit solution to the stated interpolation problems and give a number of equivalent conditions for a map to be an intertwining map. Hence, it is now possible to understand all of the RH_+^∞-module homomorphisms. Furhmann then talks about Higher Order Interpolation Problems. In the initial setup, he assumes that the zeros of the inner function $m(z)$ are distinct. An appropriate generalization is made, to deal with the situation where the zeros of $m(z)$ are not distinct.

With little doubt, the convenience of working in rational spaces is that the arguments are easier and explicit solutions can be given for the interpolation problem. Having discussed the concrete situation we move to the polar opposite situation to end our discussion.

6 Interpolation on Hilbert Spaces

The setup requires some language from functional analysis. We begin with a domain D in the complex plane. Let H be the Hilbert space of anlytic functions on D such that point evaluation at $z \in D$ gives a bounded linear functional. Define $w(z)$ to be the norm of that linear functional. We say that a sequence $\{z_j\}$ in D is a *universal interpolating sequence* for H if

$$\begin{aligned} H &\longrightarrow \ell^2 \\ f &\longmapsto \left\{ \frac{f(z_j)}{w(z_j)} \right\} \end{aligned}$$

is surjective. This setup is certainly similar to the setup that we had for our discussion of the Carleson Interpolation Problem.

The theory requires machinery that we don't have in a hurry. It is difficult to go further into the theory without the tools of functional analysis. There are other setups for function spaces such as the Paley-Weiner spaces and Bloch spaces. We mentioned that Carleson studied interpolation sequences in order to study the underlying structure of H^∞. The success of this has led others to ask: Under what circumstances does the study of universal interpolating sequences reflect the properties of the underlying function space?