Here we are going to collect some facts regarding linear operators. We assume that all vector spaces are finite-dimensional over a field $\mathbb{F}$. The rank-nullity theorem tells us that, when we are given a linear operator $T : V \longrightarrow W$, we can add up dimensions coveniently. We begin with an application of the rank-nullity theorem and prove some facts regarding short exact sequences.

Proposition 1. *Let $T : V \longrightarrow V$ be a linear operator. Then the following are equivalent:*

1. *T is bijective.*
2. *T is injective.*
3. *T is surjective.*

Proof. "$1 \Rightarrow 2$" This is true by the definition of T being a bijection.

"$2 \Rightarrow 3$" Suppose that T is injective. Then the kernel of T is trivial. The rank-nullity theorem gives us that

$$\begin{aligned}\dim V &= \dim \operatorname{Ker} T + \operatorname{rk} T\\ \dim V &= 0 + \operatorname{rk} T\\ \dim V &= \operatorname{rk} T.\end{aligned}$$

which gives us the surjectivity of T.

"$3 \Rightarrow 1$" Since T is surjective, $\operatorname{rk} = \dim V$. We again apply the rank-nullity theorem

$$\begin{aligned}\dim \operatorname{Ker} T + \operatorname{rk} T &= \dim V\\ \dim \operatorname{Ker} T &= \dim V - \operatorname{rk} T\\ &= 0\end{aligned}$$

and we conclude that the kernel of T is trivial. We conclude that T is injective and the implication that T is bijective follows immediately. □

Remark 1. We could have replaced the condition that T is bijective by the condition that G is invertible or that T is an isomorphism of V.

Short Exact Sequences

We now discuss another way to add dimensions in a convenient way. Let

$$0 \longrightarrow U \xrightarrow{T} V \xrightarrow{S} W \longrightarrow 0$$

Claim: 1. $\dim V = \dim U + \dim W$

From the rank-nullity theorem, $\dim U = \dim \operatorname{Ker} T + \operatorname{rk} T$. By the exactness of the sequence, $\operatorname{rk} T = \dim \operatorname{Ker} S$. So,

$$\dim U = \dim \operatorname{Ker} T + \dim \operatorname{Ker} S.$$

The injectivity of T gives us that $\operatorname{Ker} T = 0$ and the surjectivity of S gives us that $\operatorname{rk} S = \dim W$. Hence,

$$\begin{aligned} \dim U + \dim W &= \dim \operatorname{Ker} T + \dim \operatorname{Ker} S + \operatorname{rk} S \\ &= 0 + \dim \operatorname{Ker} S + \operatorname{rk} S \\ &= \dim V. \end{aligned}$$

The last equality is from the rank-nullity theorem.

Remark 2. A way to rephrase the claim is that

$$\dim U - \dim V + \dim W = 0.$$

Suppose $T : V \to V$ is surjective.

Then the following short sequence is exact

$$0 \longrightarrow \operatorname{Ker} T \xrightarrow{\iota} V \xrightarrow{T} V \longrightarrow 0$$

Then

$$\begin{aligned} \dim \operatorname{Ker} V - \dim V + \dim V &= 0 \\ \dim \operatorname{Ker} V &= 0 \end{aligned}$$

and therefore T is an injection, which implies that T is a bijection. With the kernel of T being trivial, one has the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} T & \xrightarrow{\iota} & V & \xrightarrow{T} & V & \longrightarrow & 0 \\ & & \| & & & & & & \\ 0 & \longrightarrow & 0 & \longrightarrow & V & \xrightarrow{T} & V & \longrightarrow & 0 \end{array}$$

Suppose that $T : V \to V$ is injective.

Then the following short sequence is exact

$$0 \longrightarrow V \xrightarrow{T} V \xrightarrow{\pi} V/\mathrm{Im}T \longrightarrow 0.$$

$$\dim V - \dim V + \dim V/\mathrm{Im}T = 0$$

$$\dim V/\mathrm{Im}T = 0$$

which gives us that $V = \mathrm{Im}T$.

That we the alternating sum of the dimesion of the vector spaces is zero is also true for the exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

by essentially the same argument given for the short exact sequence.

Applying the First Isomorphism Theorem

Carrying on, let

$$0 \longrightarrow U \xrightarrow{T} V \xrightarrow{S} W \longrightarrow 0$$

be an exact sequence. That S is surjective, we can apply the First Isomorphism Theorem $V/\operatorname{Ker} S \cong W$. By exactness, $\operatorname{Ker} S = \mathrm{Im} T \cong U$,

$$W \cong V/\operatorname{Ker} S = V/\mathrm{Im}T = V/U.$$

Furthermore, considering a basis for U and extending it to a basis for V, we can write $V = U \oplus W$, In which case T acts as a canonical injection $U \xrightarrow{\iota} U \oplus W$, and S the canonical projection $U \oplus W \to W$.

We now have three ways to view the short exact sequence:

$$0 \longrightarrow U \xrightarrow{T} V \xrightarrow{S} W \longrightarrow 0$$

$$0 \longrightarrow U \xrightarrow{\iota} V \xrightarrow{\pi} V/U \longrightarrow 0$$

$$0 \longrightarrow U \xrightarrow{\iota} U \oplus W \longrightarrow W \longrightarrow 0$$