1 Eigenvalues and Eigenvectors

We begin with some initial questions:

- What are the eigenvalues of the derivative matrix?
- Compute the eigenvalues for the matrix

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark 1. This is upper triangular and the entries on the diagonal are all the same $\lambda = 1$. The eigenvectors of T are

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So, the eigenspace associated to $\lambda = 1$ is two-dimensional.

- Consider the matrix

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Does this matrix have real eigenvalues? How about complex eigenvalues?

Definition 1. Two matrices $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{F})$ are said to be *similar* if there exists $P \in \mathrm{GL}_n(\mathbb{F})$ such that $A = P^{-1}BP$.

Lemma 1. *Similar matrices have the same eigenvalues.*

Proof. Let A and B be two similar matrices. Then there exists $P \in \mathrm{GL}_n$ such that $A = PBP^{-1}$. Well,

$$\begin{aligned} \det(A - I\lambda) &= \det(PBP^{-1} - I\lambda) \\ &= \det(P)\det(B - I\lambda)\det(P^{-1}) \\ &= \det(B - I\lambda). \end{aligned}$$

The zeros of the polynomial are uniquely determined. □

2 Diagonalization

2.1 Some Examples

Definition 2. Let $T \in \text{End}(V)$. We say that T is *diagonalizable* if there exists a basis of V consisting of eigenvalues of T.

Example 1. The matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is not diagonalizable as a real operator. This is becuase

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1.$$

Example 2. The matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is diagonalizable. One can easily compute that the eigenvalues are $\lambda = 1$ and $\lambda = -1$. One then computes the eigenspaces associated to these values:

$$E_{\lambda=1} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \text{ and } E_{\lambda=-1} = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle.$$

Example 3. Diagonalize the matrix in the previous example. Let $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$. We can find P^{-1} by row operations; $P^{-1} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{pmatrix}$. Then

$$\begin{aligned} P \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} P^{-1} &= P \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{pmatrix} \\ &= P \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

2.2 Some Theoretical Considerations

Let $T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We find that its characteristic polynomial is

$$\det(T - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3$$

So its eigenvalue is $\lambda = 1$.

Q: How many power of $1-\lambda$ do we need to get T to vanish?

A: 2

Example 4. Let $S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. We've seen that this is diagonalizable. It has characteristic polynomial $p(\lambda) = (\lambda - 1)(1 - \lambda^2)$. One would note that S will vanish on $1 - \lambda^2$.

Proposition 1. *If an operator T is diagonalizable then its minimal polynomial splits into distinct linear factors.*

Theorem 1. *Let V be a finite-dimensional vector space and $T \in \mathrm{End}(V)$ and $\lambda \in \mathbb{F}$. The following are all equivalent:*

1. *λ is an eigenvalue for T.*

2. *The operator $T - \lambda I$ is singular.*

3. *$\det(T - \lambda I) = 0$.*

Remark 2. We make the following observation in order to continue our discussion. Suppose that $T \in \mathrm{End}(V)$ and $f \in \mathbb{F}[z]$. Then $f(T) \in \mathrm{End}(V)$. It is using this that one obtains an action of the PID $\mathbb{F}[x]$ on a finite-dimensional vector space V.

Lemma 2. *Suppose that $Tv = \lambda v$. Then $f(T)v = f(\lambda)v$.*

Proof. We write $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and using the remark we want to consider the operator $f(T) = a_n T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0 I$. For any given j,

$$a_j T^j v = a_j T^{j-1} \lambda v = a_j \lambda^j v.$$

Using this idea on every monomial and the linearity of T we get the statement. $\square$

Theorem 2. *Let* $T \in \mathrm{End}(V)$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *the distinct eigenvalues of* T*. Suppose* $W_i = \mathrm{Null}(T - \lambda_i I)$*. The following are all equivalent:*

1. T *is diagonalizable.*
2. *The characteristic polynomial of* T *is* $p(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_k)^{e_k}$*, where* $e_1 + e_2 + \cdots + e_k = \dim V$ *and* $\dim W_i = e_i$*.*
3. $\dim V = \sum_{i=1}^{k} \dim W_i$*.*